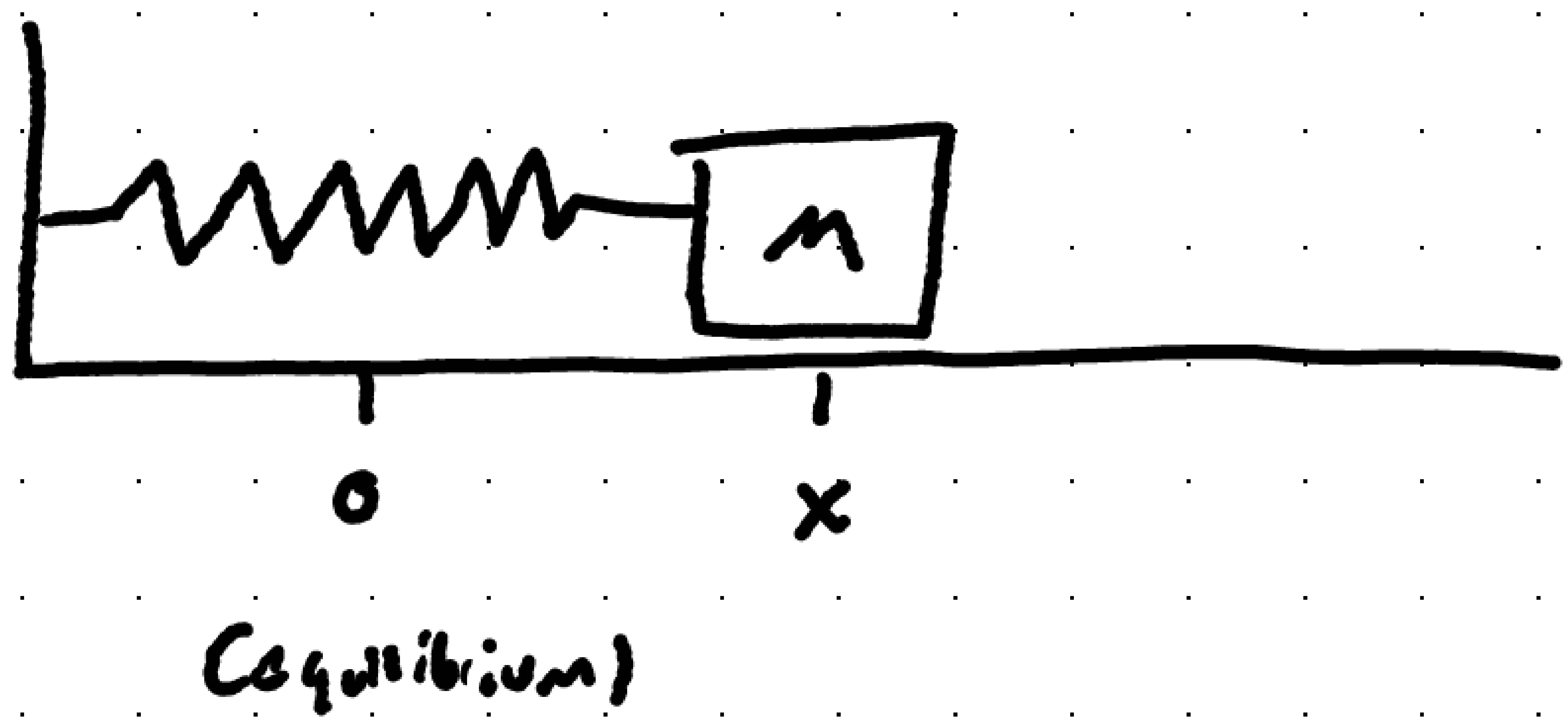
Application: Vibrating Strings



Newton's 2^{nd}: $F = ma$ $\quad a = \dfrac{d^2 x}{dt^2}$

Let $x(t)$ be the displacement of mass from the equilibrium.

Hooke's Law:

F acting on mass, due to spring, is proportional to $x(t)$ (displacement from equilibrium)

$F = -kx, \quad k > 0$

Plugging both in ends up with:

$$M \frac{d^2 x}{dt^2} = -Kx$$

or

$$M\ddot{x}(t) + Kx(t) = 0$$

Solve Homogenous:

$Mr^2 + K = 0$, $r^2 = -\frac{K}{m}$, $r = \pm\sqrt{-\frac{K}{m}} = \pm\sqrt{\frac{K}{m}}\, i$

and just for notation, let's say

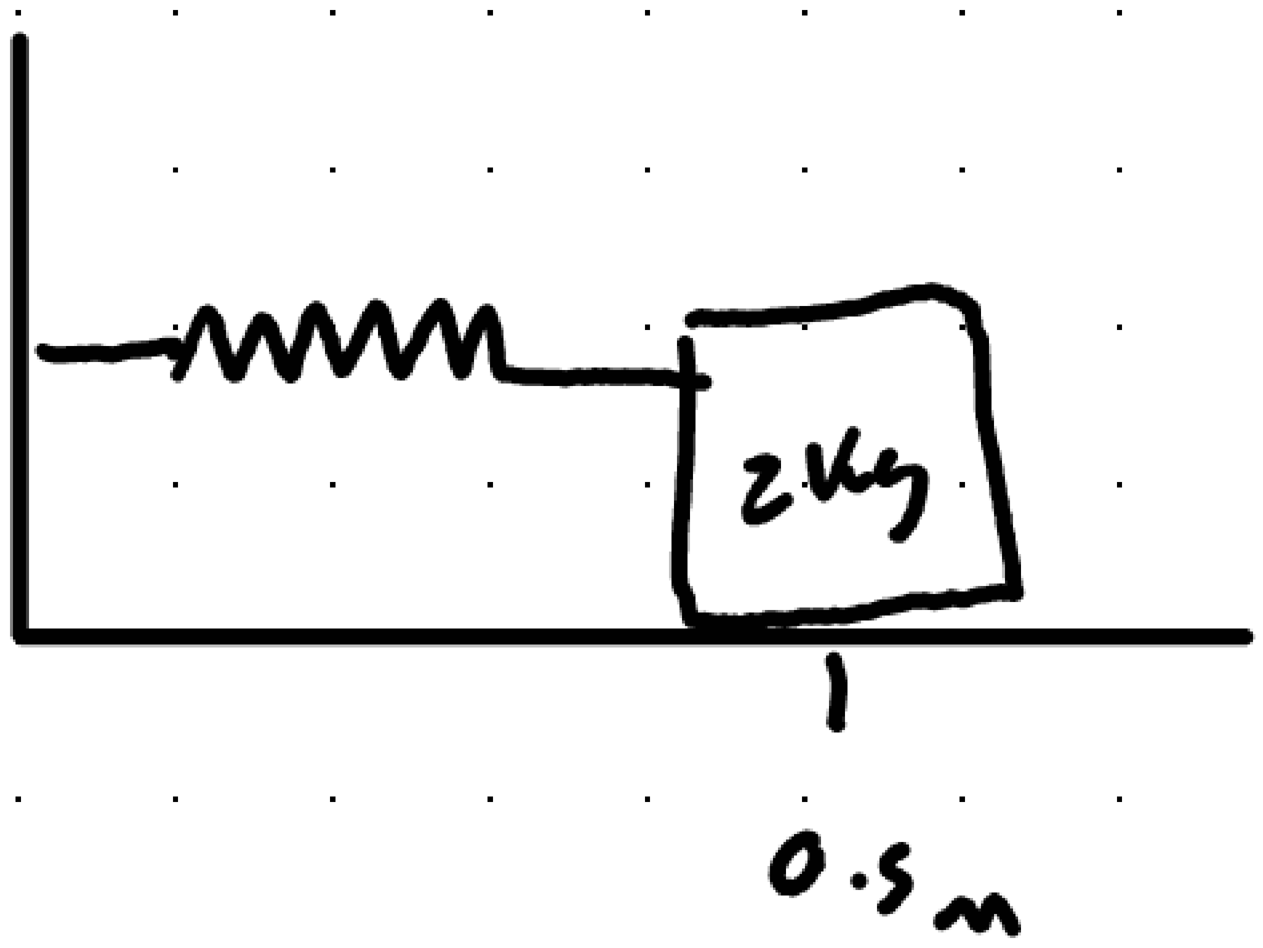
$$r = \pm \omega i$$

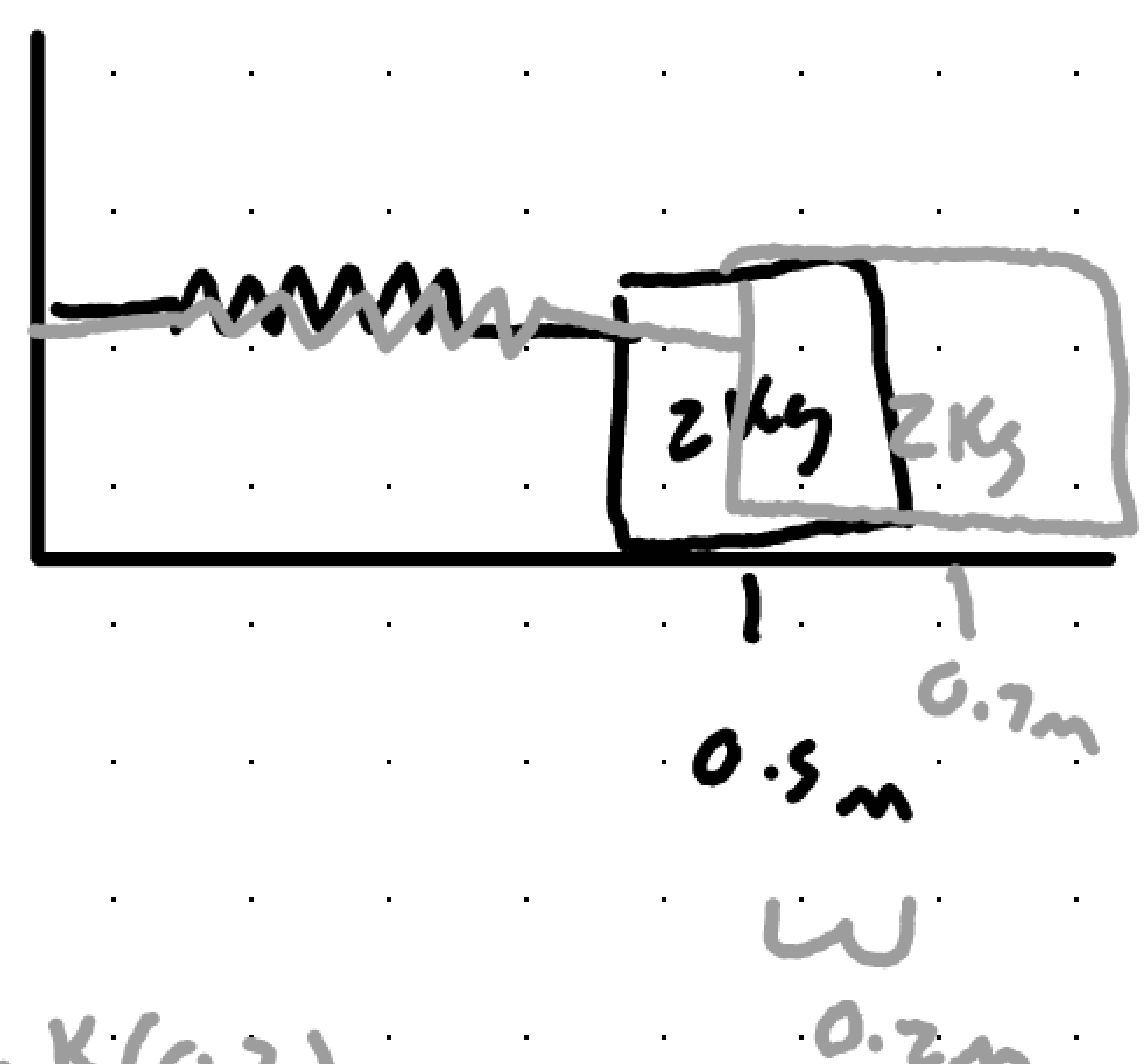
Solution

$$x(t) = C_1 \cos\omega t + C_2 \sin\omega t, \quad \omega = \sqrt{\frac{K}{m}}$$

Example:

A Spring with a mass of 2 kg has a natural length 0.5m. A force of 25.6 N is required to stretch it to a length of 0.7m. The spring is then released with an initial velocity of 0. Find the position of the mass at any time t.





$F = -kx$

$-25.6 = -k(0.2)$

$\boxed{k = 128}$

$$2\frac{d^2x}{dt^2} + 128x = 0$$

$$x(t) = C_1\cos 8t + C_2\sin 8t$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{128}{2}} = 8$$

$$2\frac{d^2x}{dt^2} + 128x = 0$$

Initial Conditions

$x(0) = 0.2$ distance

$x'(0) = 0$ velocity

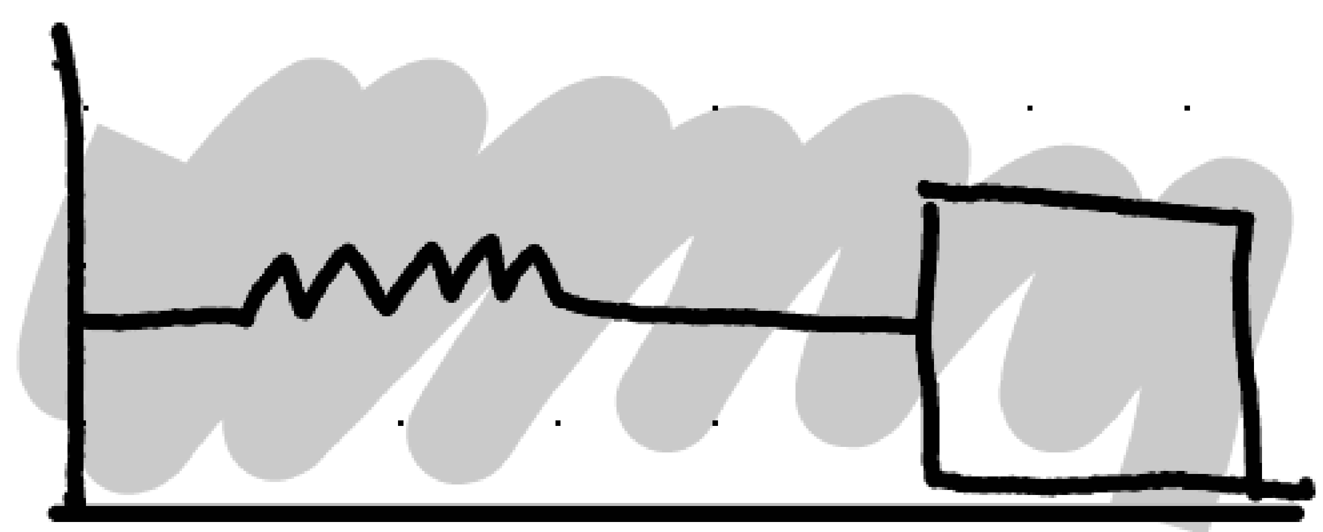
$$x(t) = C_1\cos 8t + C_2\sin 8t$$

$$x(0) = C_1 = 0.2$$

$$\dot{x}(0) = -8C_1\sin 8t + 8C_2\cos 8t\Big|_{t=0} = 8C_2 = 0$$

$$\boxed{x(t) = 0.2\cos 8t}$$

Damped Vibrations



Say, in damping fluid

Damping force: Proportional to velocity, $\frac{dx}{dt}$

$\downarrow$

$-c\frac{dx}{dt}$

$$m\frac{d^2x}{dt^2} = \underset{\text{Force}}{\text{Restoring}} + \underset{\text{Force}}{\text{Damping}} = -Kx - c\frac{dx}{dt}$$

$$\boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + Kx = 0}$$

Solve

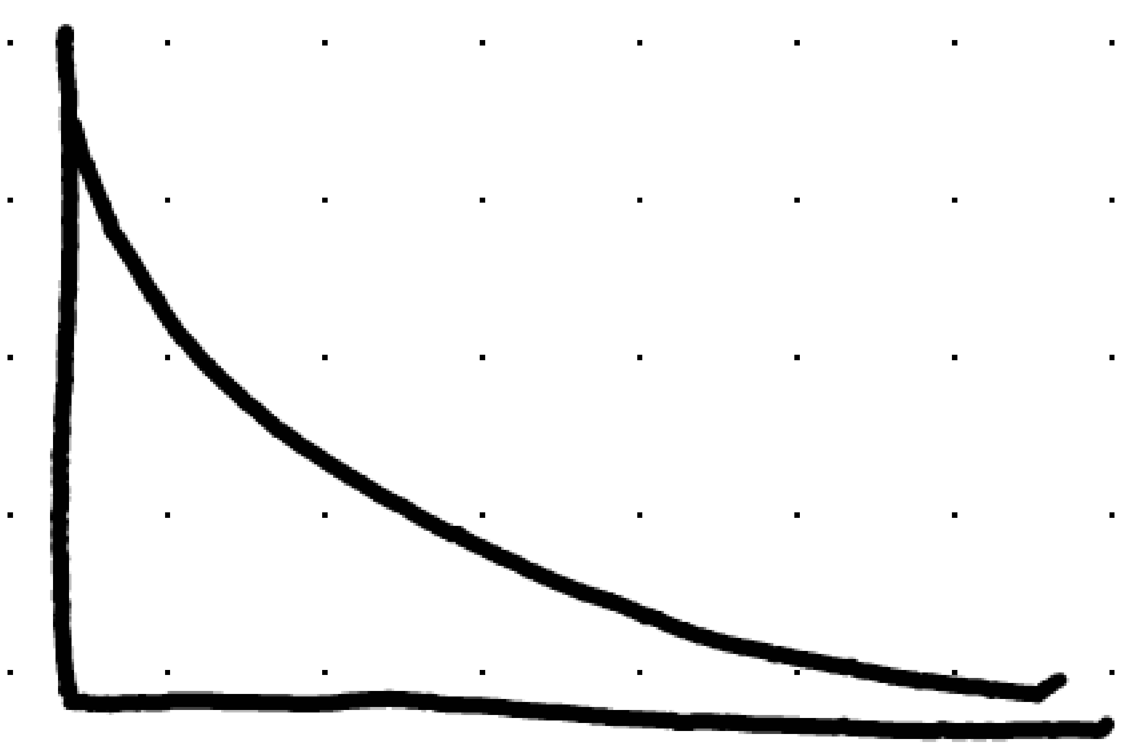
$$mr^2 + cr + K = 0$$

$$r = \frac{-c \pm \sqrt{c^2 - 4mK}}{2m}$$

Case I $\quad c^2 > 4mk \quad$ (overdamping)

Two real roots to the char eq

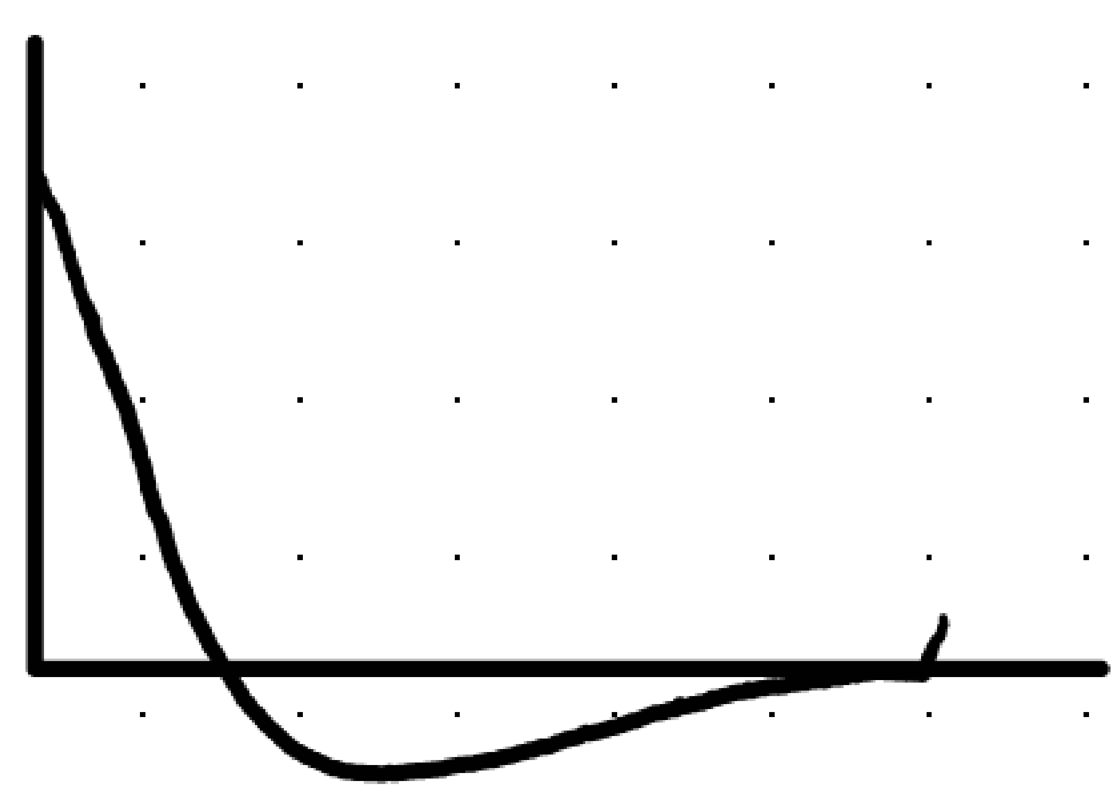
$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$



Case II $\quad c^2 - 4mk = 0$

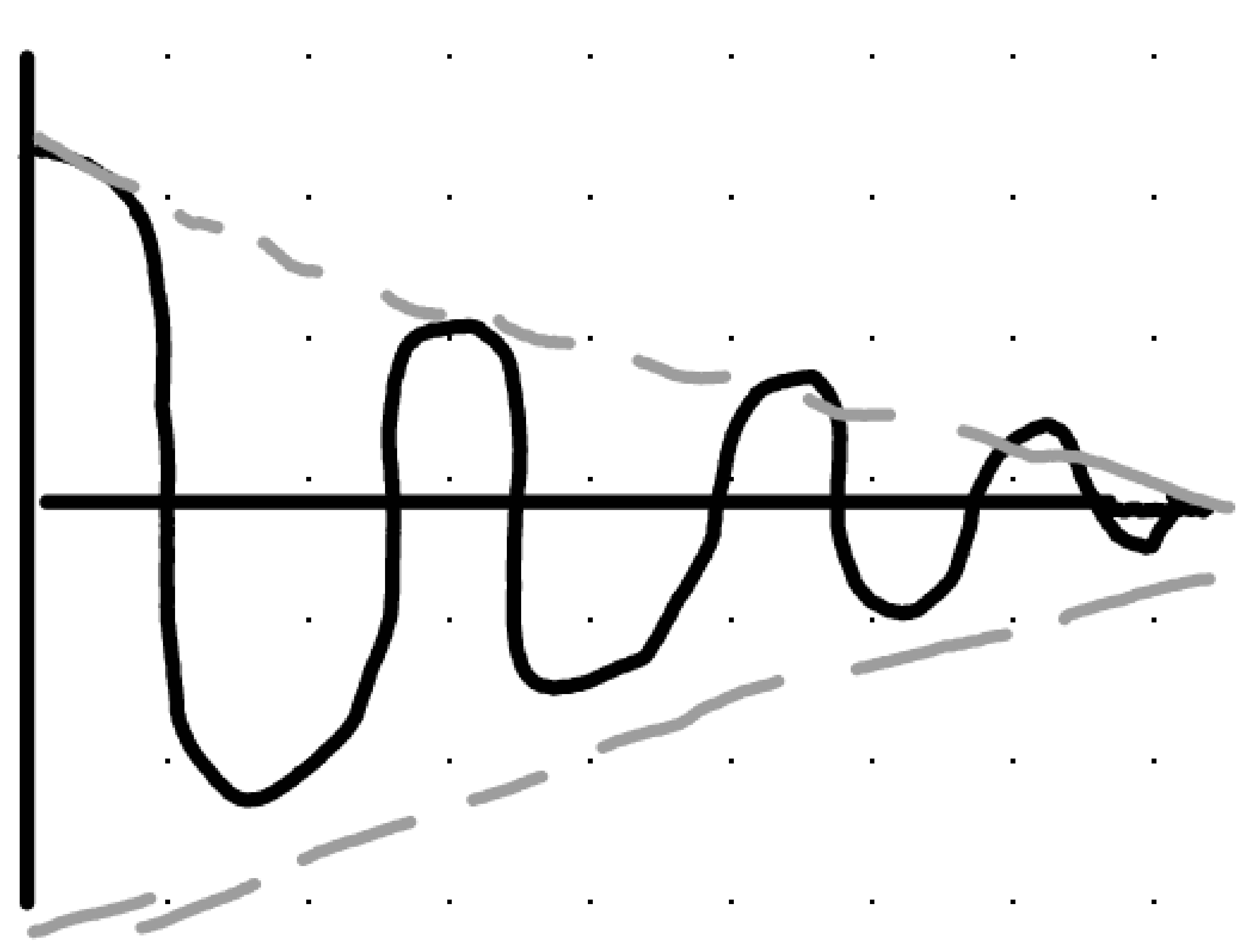
One root: $r = \frac{-c}{2m}$ $\quad$ Critical damping

$$x(t) = c_1 e^{\frac{-c}{2m} t} + c_2 t e^{\frac{-c}{2m} t}$$



Case III $\quad c^2 - 4mk < 0 \quad$ Underdamping

$$x(t) = e^{\frac{-c}{2m} t}(c_1 \cos \omega t + c_2 \sin \omega t)$$

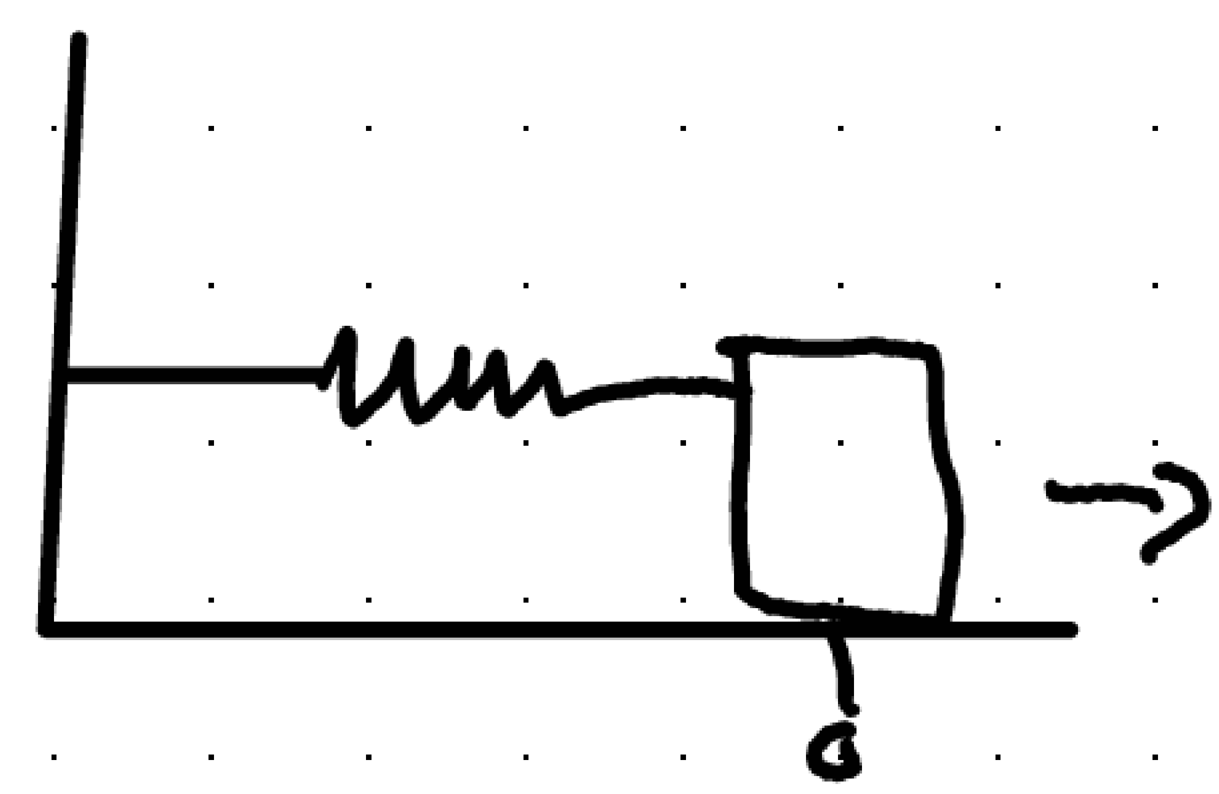


Exponential decay on oscillating wave

Example: Suppose the spring from the previous example is immersed in a damping fluid with a coefficient $c = 40$. (Recall, $k = 128$, $m = 2$)

Find the position of the mass at any time t if it starts from equilibrium and is given a push with initial velocity 0.6 m/s

$x(0) = 0$, $x'(0) = 0.6$ m/s



$$2\frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 128x = 0$$

$$\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 64x = 0$$

$$r^2 + 20r + 64 = 0$$

$$(r+4)(r+16)$$

$r = -4$, $r = -16$

$$x(t) = c_1 e^{-4t} + c_2 e^{-16t}$$

Apply IC's

$x(0) = 0$

$\dot{x}(0) = 0$

$x(0) = c_1 + c_2 = 0$

$c_2 = c_1$

Forced Vibrations

External Force: $F(t)$

$$m\frac{d^2x}{dt^2} = \text{restoring} + \text{damping} + \text{external forces}$$

$$= Kx - c\frac{dx}{dt} + F(t)$$

$$\boxed{m\ddot{x} + c\dot{x} + Kx = F(t)}$$ Non-Homogeneous

Common Situation

External Force is oscillating: $F(t) = F_0 \cos W_0 t$

(Where $W_0 \neq W = \sqrt{\frac{K}{m}}$)
(Frequency)

If $c=0$

$$m\ddot{x} + Kx = F_0 \cos wt$$

Can express solution as:

$$x(t) = c_1\cos wt + c_2\sin wt + \frac{F_0}{m(W^2 - W_0^2)}\cos(W_0 t)$$

If it turns out that the driving force frequency is the same as the natural frequency

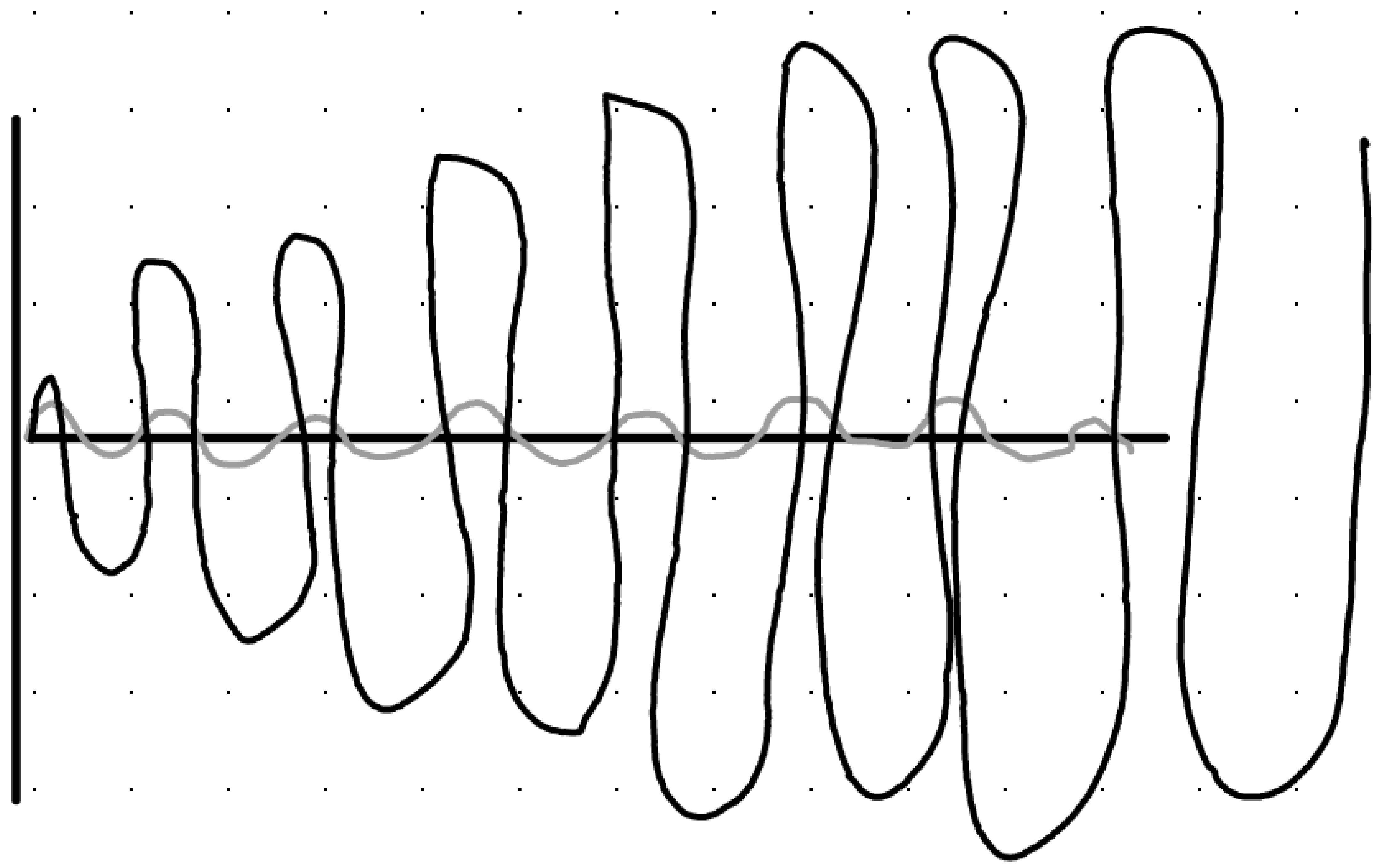
$\boxed{\omega_0 = \omega}$

We can show

Dupe shows up!

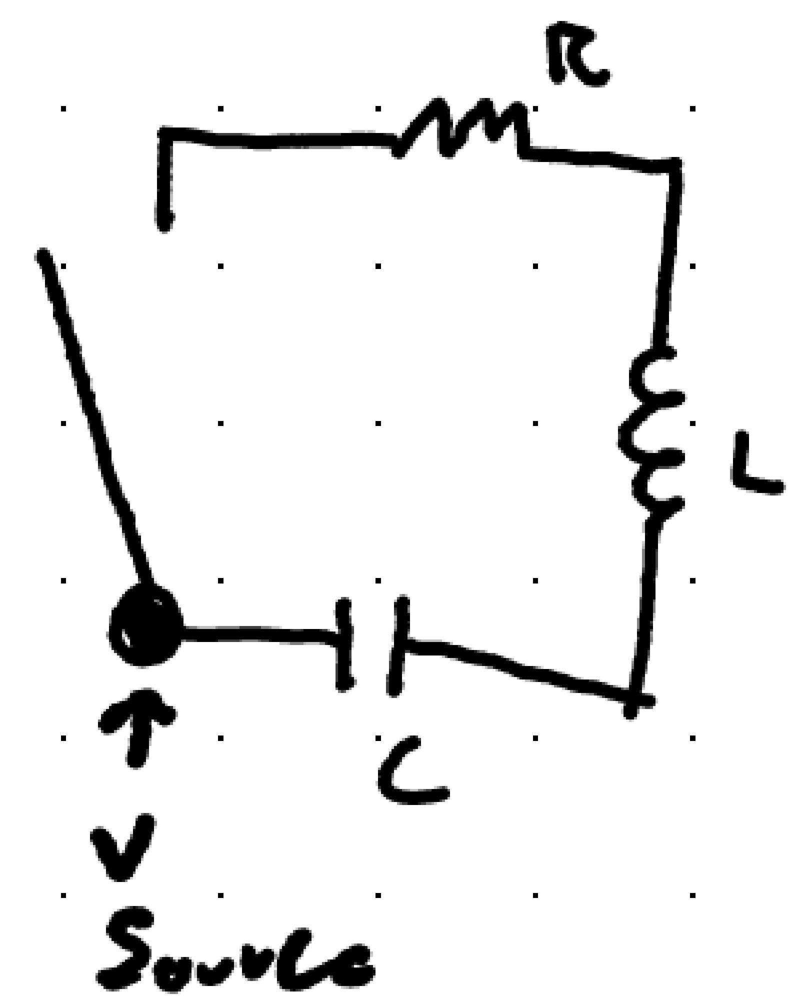
$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{2m\omega} t \sin \omega t$$

When the driving force frequency matches the natural frequency, you get resonance



Picture pushing on a swing in the absence of any damping force...

Electric Circuits



Electromotive Force, $E(t)$ (e.g battery)

Resistor R (resistance)
Inductor L (inductance)
Capacitor C (capacitance)

$Q(t)$: Charge on capacitor at time t.

$I(t)$: Current at time t.

$$I(t) = \frac{dQ}{dt}$$

$$E(t) = L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q$$

Example:

Find $Q(t)$ and current ($I(t)$) at time t if $R = 40\,\Omega$, $L = 1\,H$, $C = 16 \times 10^{-4}\,F$, $E(t) = 100\cos 10t$ and initial charge and current are both zero.

$$1\ddot{Q} + 40\dot{Q} + \frac{1}{1.6\times 10^{-3}}Q = 100\cos 10t$$

$$\boxed{\ddot{Q} + 40\dot{Q} + 625Q = 100\cos 10t}$$

$$r^2 + 40r + 625 = 0$$

$$r = \frac{-40 \pm \sqrt{1600 - 2500}}{2}$$

$$\boxed{r = -20 \pm 15i}$$

$$Q(t) = e^{-20t}(C_1\cos 15t + C_2\sin 15t)$$

Next, find $Q_p(t)$

$\boxed{\text{Guess}}$ No Dupe!

$$Q_p(t) = A\cos 10t + B\sin 10t$$
$$\dot{Q}_p(t) = -10A\sin 10t + 10B\cos 10t$$
$$\ddot{Q}_p(t) = -100A\cos 10t - 100B\sin 10t$$

Plug into $Q_p'' + 40Q_p' + 625Q_p$

$= \dots$

$=$

$$(525A + 400B)\cos 10t + (-400A + 525B)\sin 10t = 100\cos 10t$$

Determine A, B:

cos:

$525A + 400B = 100$

sin

$-400A + 525B = 0$

$$\boxed{A = \frac{84}{697}}$$

$$\boxed{B = \frac{64}{697}}$$

$$Q_p(t) = \frac{84}{697}\cos 10t + \frac{64}{697}\sin 10t)$$

So:

$$Q(t) = \frac{84}{697}\cos 10t + \frac{64}{697}\sin 10t) + e^{-20t}(C_1\cos 15t + C_2\sin 15t)$$

Apply IC's

$Q(0) = 0, \quad I(0) = 0$

$\hookrightarrow Q'(0) = 0$

$$Q(0) = (C_1 + 0) + \frac{84}{697} + 0 = 0, \quad C_1 = \frac{-84}{697}$$

$$Q'(0) = \ldots = -20C_1 + 15C_2 + \frac{640}{697} = 0$$

$$C_2 = -\frac{464}{2091}$$

$$\boxed{Q(t) = \underbrace{\frac{84}{697}\cos 10t + \frac{64}{697}\sin 10t)}_{\substack{\text{Steady State Solution}\\ \text{(Always sticks around)}}} + \underbrace{e^{-20t}\left(\frac{-464}{2091}\cos 15t + \frac{-84}{697}\sin 15t\right)}_{\substack{\text{Transient Solution}\\ \text{(eventually goes away)}}}}$$