Consider the system

$$\dot{y}_1 = y_1$$
$$\dot{y}_2 = y_1 - y_2$$

with initial conditions $y_1(0) = 1, \; y_2(0) = 2$

Let's Solve

$\dot{y}_1 = y_1 \implies$ Need a function who's derivative is itself.

$\hookrightarrow$ $y_1(x) = C_1 e^x$

Apply IC's

$y_1(0) = 1$: $C_1 = 1$

Plug this into Second Equation

$$\dot{y}_2 = e^x - y_2$$

$$\dot{y}_2 + y_2 = e^x$$

Say we now want to solve for y_2

Integrating factor method

$\dot{y}_2 + 1y_2 = e^x$

$p(x) = 1$, so take $e^{\int p(x)dx} = e^{\int 1 dx} = e^x$

Now, we multiply everything by e^x

$e^x\dot{y}_2 + e^x y_2 = e^x e^x$

$(e^x y_2)' = e^{2x}$

Expanded Product Rule

Then, differentiate both sides...

$e^x y_2 = \frac{1}{2}e^{2x} + C_2$

Constant gets put in place

Then, divide e^x out

$$\boxed{y_2(x) = \frac{1}{2}e^x + C_2 e^{-x}}$$

General Solution

$$y_1(x) = c_1 e^x$$

$$y_2(x) = \frac{c_1}{2} e^x + c_2 e^{-x}$$

If we hadn't solved for c_1 way back.

Solution Satisfying IC's

$y_1(0) = 1 \quad y_2(0) = 2$

$$y_2(0) = \frac{1}{2} + c_2 = 2$$

$$c_2 = \frac{3}{2}$$

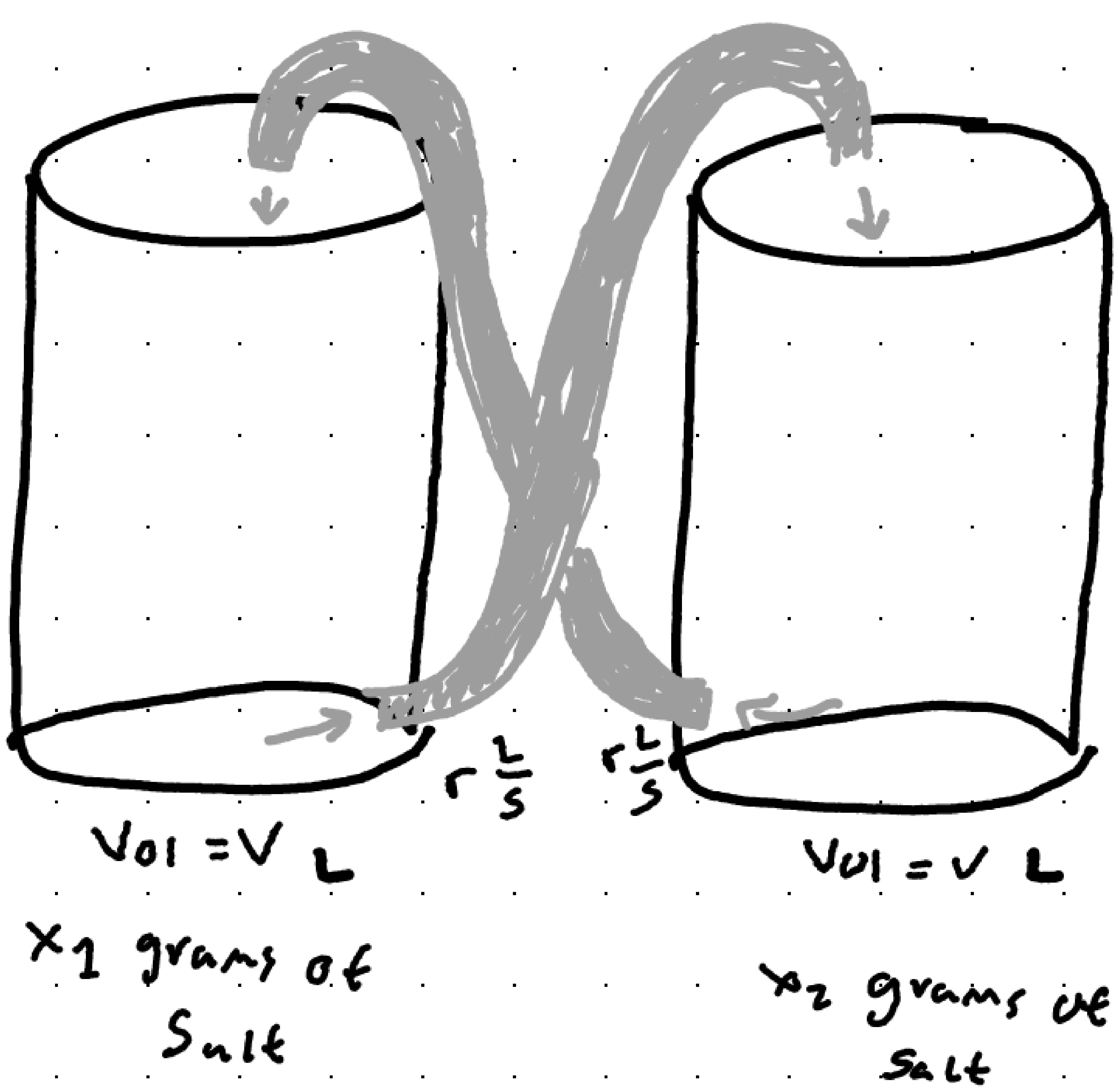
So,

$$\boxed{\begin{aligned} y_1(x) &= e^x \\ y_2(x) &= \frac{1}{2} e^x + \frac{3}{2} e^x \end{aligned}}$$

Physical Systems

Example

Suppose we have two salt water tanks, each with a different concentration of salt, with salt water flowing back and forth between them. We are interested in how the amount of salt in each tank is changing over time. (We'll assume the salt water is always thoroughly mixed))



Water flows from tank 1 to tank 2 & from tank 2 to tank 1 at a constant rate of

$$r \frac{L}{s}$$

$$\text{Concentration} = \frac{x_n}{V} \frac{g}{L}$$

What's the rate of change of $x_1(t)$? i.e. what's $x_1'(t)$?

$x_1'(t)$ = "rate in" − "rate out"

$x_1(t)' =$ "rate in" $-$ "rate out"

$$= \left(r\ \tfrac{L}{s}\right)\cdot\left(\tfrac{x_2}{V}\ \tfrac{g}{L}\right) - \left(r\ \tfrac{L}{s}\right)\cdot\left(\tfrac{x_1}{V}\ \tfrac{g}{L}\right)$$

$$\boxed{x_1' = \frac{r}{V}(x_2 - x_1)}$$

Similarly,

$$\boxed{x_2' = \frac{r}{V}(x_1 - x_2)}$$

System:

$$x_1' = \frac{r}{V}(x_2 - x_1)$$
$$x_2' = \frac{r}{V}(x_1 - x_2)$$

First order, Linear, homogeneous, autonomous

↓

Constant coefficient Volume and Rate are constant.

Note: If $x_1(0) = x_2(0) = A$
(salt is the same in tank)

then a solution is given by: $x_1(t) = A$, $x_2(t) = A$

Also,

If $x_1(0) = A$, $x_2(0) = B$, then

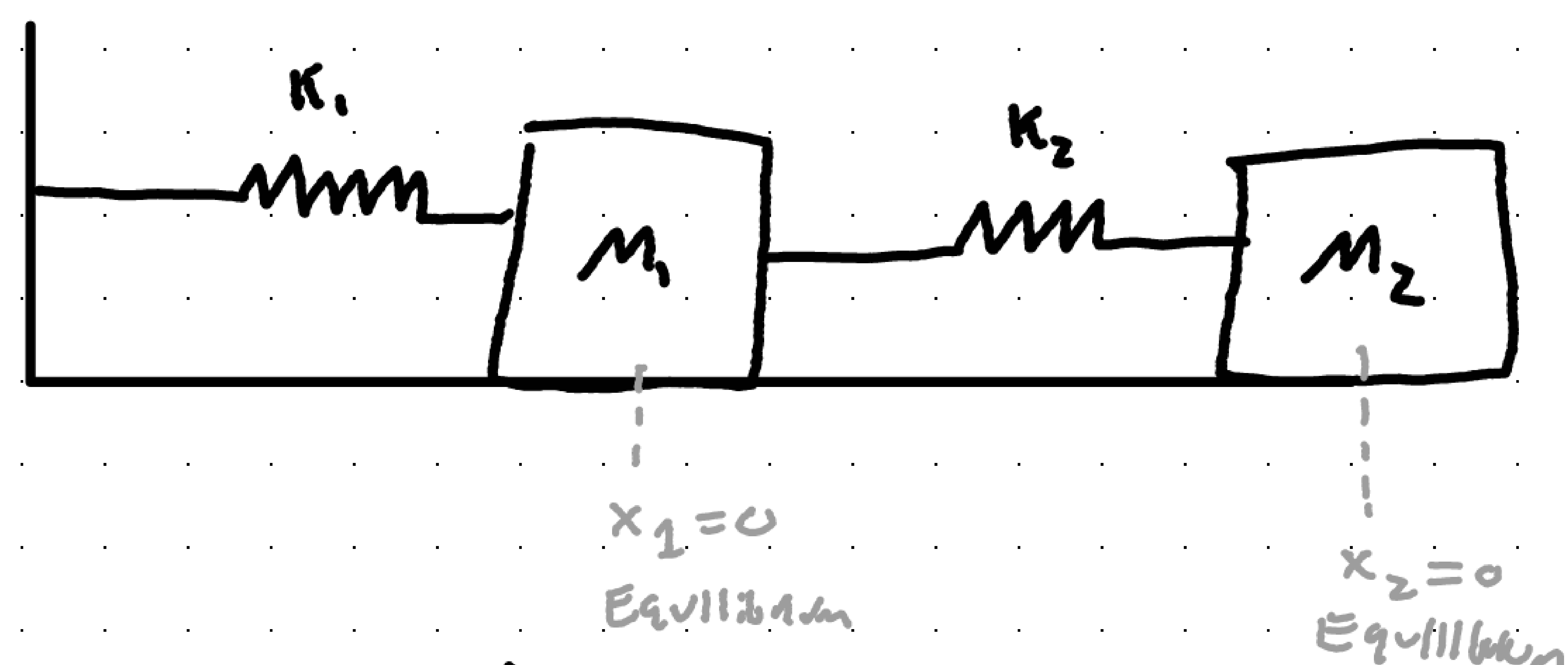
$A + B$ is constant, and eventually

$$x_1(t) \approx \frac{A+B}{2}$$

$$x_2(t) \approx \frac{A+B}{2}$$

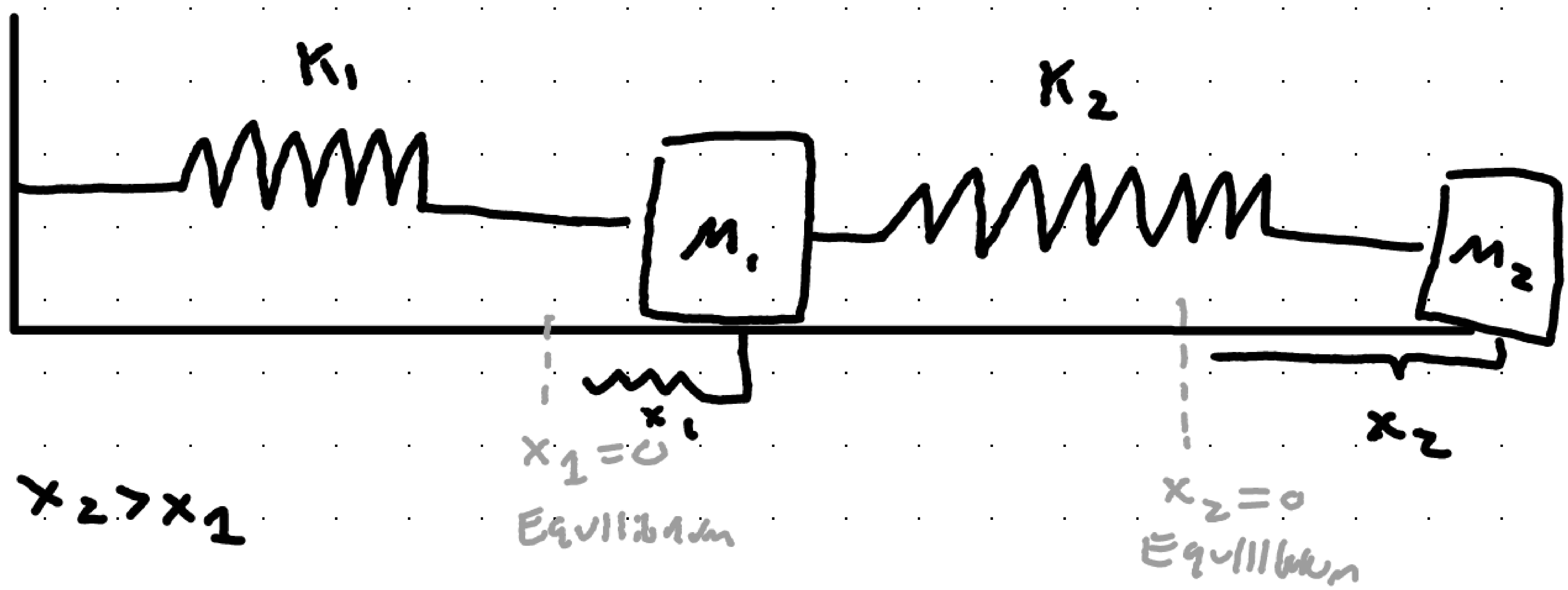
Example:

Consider two masses, M_1 & M_2, on two springs, sliding on frictionless plane. Let $x_1(t)$ denote the displacement from equilibrium of M_1 and $x_2(t)$ the displacement from equilibrium of M_2



Newton's 2^{nd}, $F = ma$

Scenario



$x_2 > x_1$

x_2 is quite stretched!

Force on M_2: Hooke's Law

$$\vec{F} = -K_2(x_2 - x_1)$$

So,

$$\boxed{M_2 x_2'' = -K_2(x_2 - x_1)}$$

Force on M_1: Hooke's Law Again

$$\vec{F} = -K_1 x_1 + K_2(x_2 - x_1)$$

So,

$$\boxed{M_1 x_1'' = -K_2(x_2 - x_1)}$$

Second Order System:

$$\boxed{\begin{aligned} M_1 x_1'' &= -(K_1 + K_2)x_1 + K_2 x_2 \\ M_2 x_2'' &= K_1 x_1 - K_2 x_2 \end{aligned}}$$

We're going to mostly consider first order systems (although we will revisit the second order spring mass system).

This isn't as restrictive as it seems, because we can convert higher order DE's or higher order systems, as follows:

Ex: $\ddot{y} + 2\dot{y} + 3y = 0$

Introduce new variables y_1, y_2:

$y_1 = y$, so $\dot{y}_1 = \dot{y} = y_2$

$y_2 = \dot{y}$

$$\dot{y}_2 = \ddot{y} = -2\dot{y} - 3y$$
$$= -2y_2 - 3y_1$$

System would be:

$$\boxed{\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -3y_1 - 2y_2 \end{aligned}}$$

Example:

$$m_1 \ddot{x}_1 = -K_1 x_1 + K_2 (x_2 - x_1)$$
$$m_2 \ddot{x}_2 = -K_2 (x_2 - x_1)$$

Introduce new Variables: U_1, U_2, U_3, U_4

$U_1 = x_1$ $\qquad \dot{U}_1 = \dot{x}_1 = U_2$

$U_2 = \dot{x}_1$ $\qquad \dot{U}_2 = \ddot{x}_1 = -\frac{K_1}{m_1} x_1 + \frac{K_2}{m_1}(x_2 - x_1)$

$U_3 = x_2$ $\qquad = \frac{-K_1}{m_1} U_1 + \frac{K_2}{m_1}(U_3 - U_1)$

$U_4 = \dot{x}_2$

$$\dot{U}_3 = \dot{x}_2 = U_4$$
$$\dot{U}_4 = \ddot{x}_2 = -\frac{K_2}{m_2}(x_2 - x_1)$$
$$= \frac{-K_2}{m_2}(U_3 - U_1)$$

$$\dot{U}_1 = U_2$$
$$\dot{U}_2 = \frac{-(K_1 + K_2)}{m_1} U_1 + \frac{K_2}{m_1} U_3$$
$$\dot{U}_3 = U_4$$
$$\dot{U}_4 = \frac{K_2}{m_2} U_1 - \frac{K_2}{m_2} U_3$$

It's convenient to express linear systems of ODE's in matrix vector form.

Let's revisit that earlier example we saw.

$$\dot{x} = 2x + 3y$$
$$\dot{y} = x - 4y$$

Unknowns $x(t)$, $y(t)$

Let's put x, y in a vector: $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

So, $\dot{\vec{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$

$$\begin{bmatrix} 2x + 3y \\ x - 4y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\boxed{\vec{x}' = A\vec{x}}$$

Homogeneous system

If we have n solutions, $\vec{x}_1, \vec{x}_2, \vec{x}_3 \dots \vec{x}_n$ and put them in a matrix X, we can determine linear independence by taking the determinant of X. (This determinate is given by a special name: The Wronskian) If the Wronskian is non-zero, the solutions are linearly independent and the matrix X is called a fundamental matrix solution

— — —

e.g

$$\dot{x}_1 = x_1 \,, \quad \dot{x}_2 = x_1 - x_2$$

$$\vec{x}_1(t) = \begin{bmatrix} e^t \\ \frac{e^t}{2} \end{bmatrix} \qquad \vec{x}_2(t) = \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$$

$$X = \begin{bmatrix} e^t & 0 \\ \frac{e^t}{2} & e^t \end{bmatrix}$$

$$\begin{aligned} W &= e^t \cdot e^{-t} - 0 \cdot \frac{e^t}{2} \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

$1 \neq 0$

Therefore $\vec{x}_1$ & $\vec{x}_2$ are L.I, and X is a fundamental matrix solution