To gain some geometric insight, and deepen our understanding, let's restrict ourselves to first order linear systems, with two dependent variables.

Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The Eigenvalues of the Matrix are $\lambda = 1, 2$, with the corresponding Eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so we can write the general solution as $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 e^t + c_2 e^t \\ c_2 e^t \end{bmatrix}$

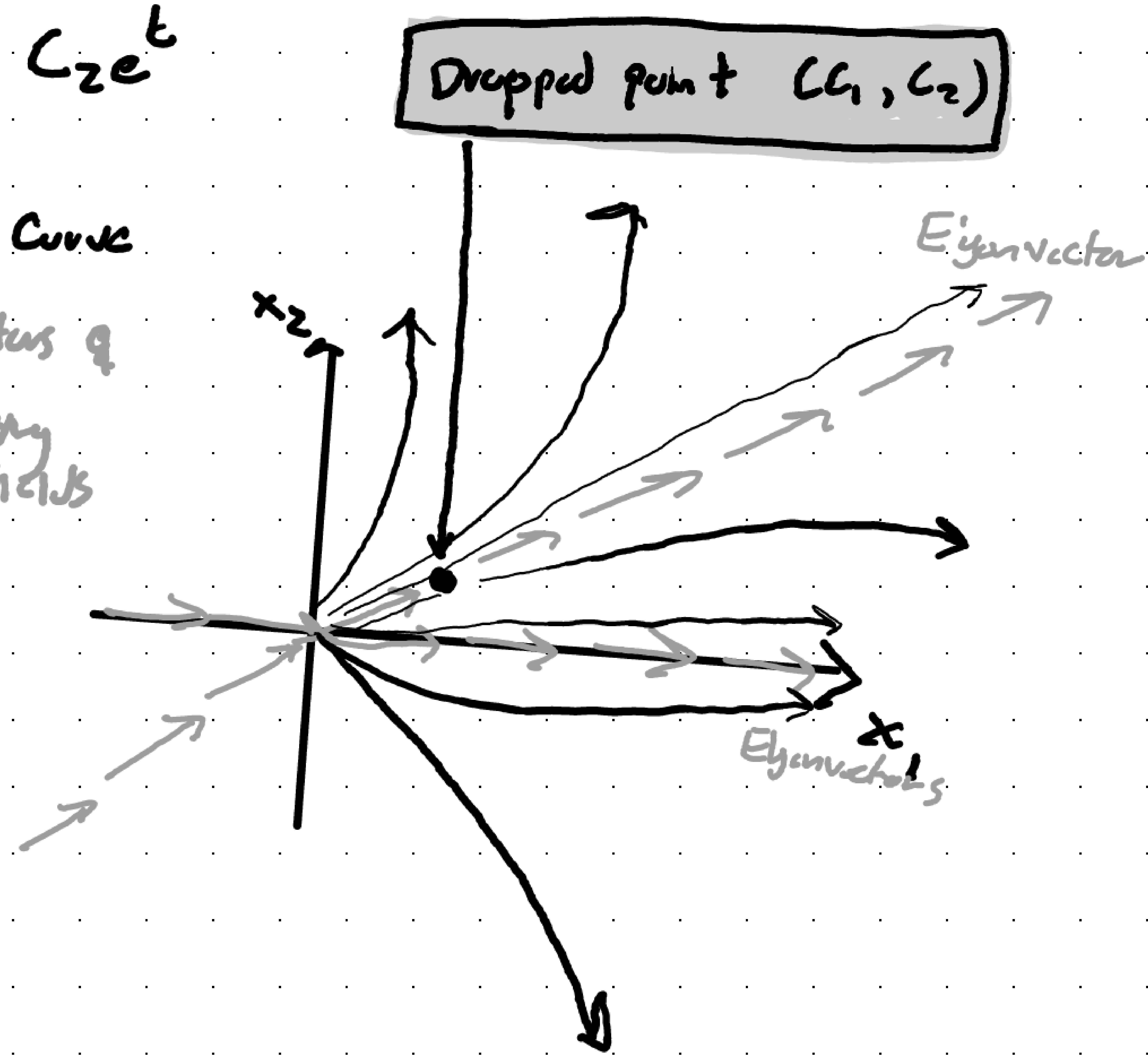
Let's plot $x_1(t)$ & $x_2(t)$

$$x_1(t) = c_1 e^t + c_2 e^t$$

$$x_2(t) = c_2 e^t$$

■ - Solution Curve

▨ - Eigenvectors & accompanying vector fields



$\boxed{t=0}$

$$x_1 = c_1 + c_2$$

$$x_2 = c_2$$

The Eigenvectors are stable axes!

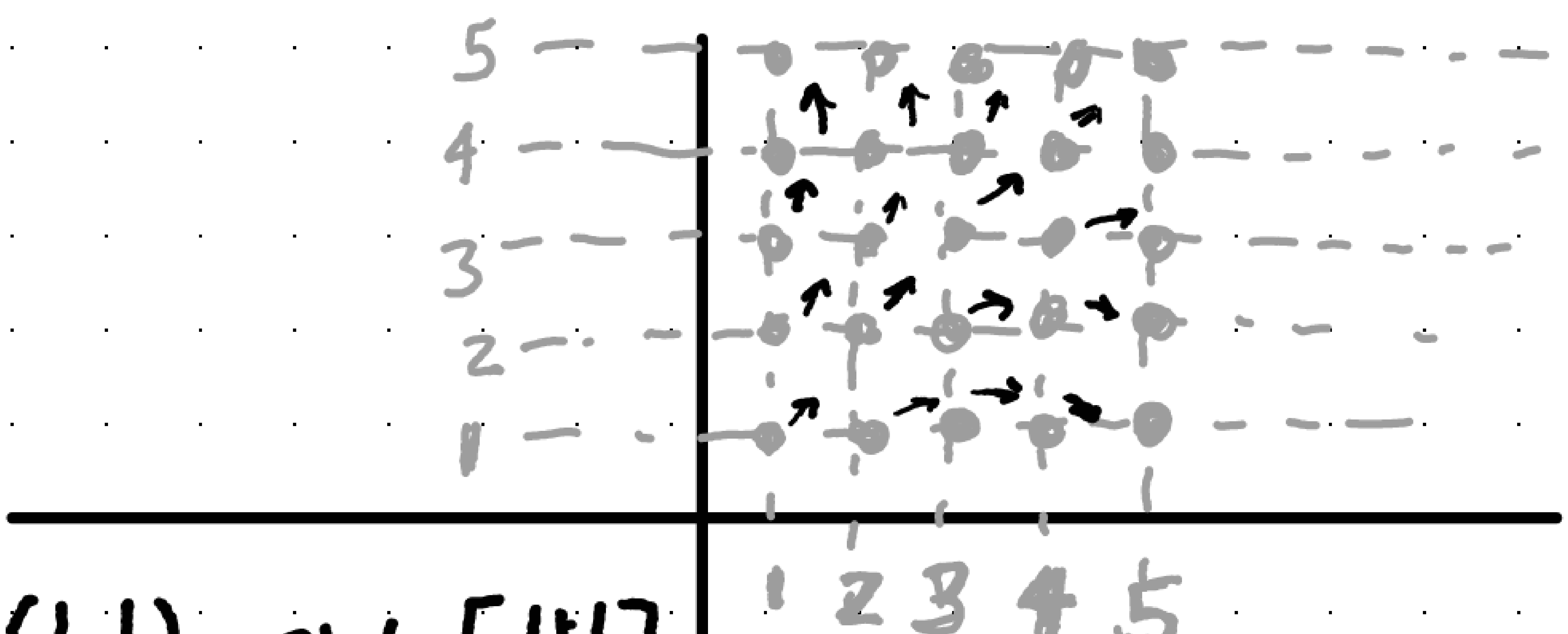
This system is an "unstable Node" or "Source" as both eigenvalues are positives. Vectors leave origin.

We can think of the system, in particular the matrix A, as giving rise to a "vector field" in the plane. At each point (a,b) in the plane, plot the vector obtained by multiplying by the <u>matrix A</u>.

<u>Ex:</u> $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

If $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$, consider "fixed points" (a,b)

$$A\vec{x} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ 2b \end{bmatrix}$$



At $(1,1)$, plot $\begin{bmatrix} 1+1 \\ 2 \end{bmatrix}$ unit vector

At $(1,2)$, plot $\begin{bmatrix} 1+2 \\ 4 \end{bmatrix}$ unit vector

... etc

When we are finding Solutions to a Linear System of First order DE'S, with a constant Matrix A, we're looking for functions $x_1(t)$ & $x_2(t)$, So that the tangent vector (the direction) is given by the Matrix multiplied by a Position vector.

That's why our Solution curves from the previous examples line up with the vector field.

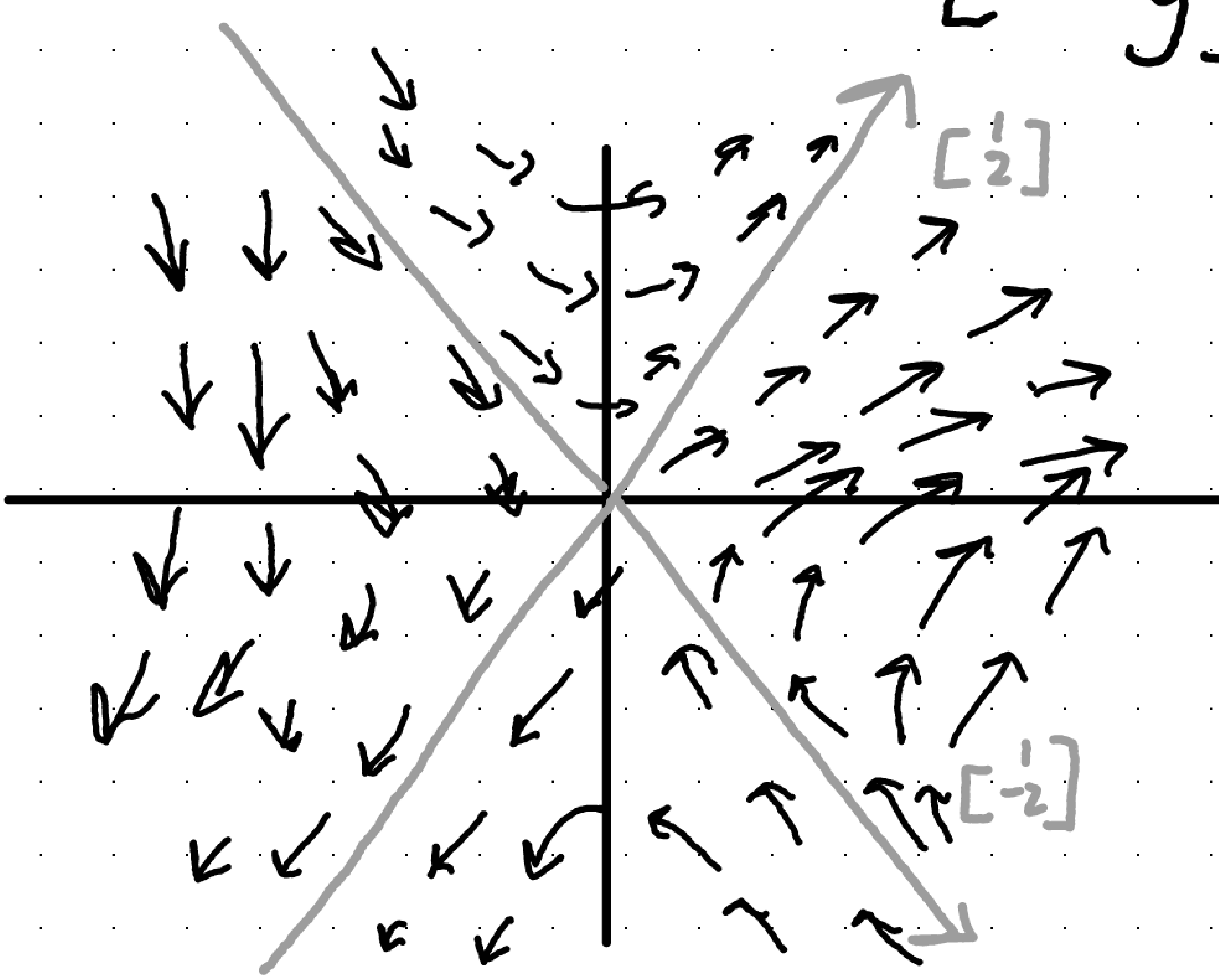
Example:

Consider the system

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x}$$

The Eigenpairs are $3, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ & $-1, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ 4x + y \end{bmatrix}$$



Note:

Positive valued Eigenvalues attached to a Eigenvector indicate that the arrows will AWAY from zero, where negatives will point towards.

$$\vec{x}(t) = \underbrace{c_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}} + c_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

At a large time t, e^{3t} will get really big, while e^{-t} gets really small. This explains why our arrows prefer the positive line to the negative one.

This is an example of a Saddle

- One Eigenvalue is positive, one is negative
- Eigenvalues are distinct.

You can also have "Stable Nodes" or "Sinks". All the vector field components face zero, and both Eigenvalues are negative.

Complex Eigenvalues

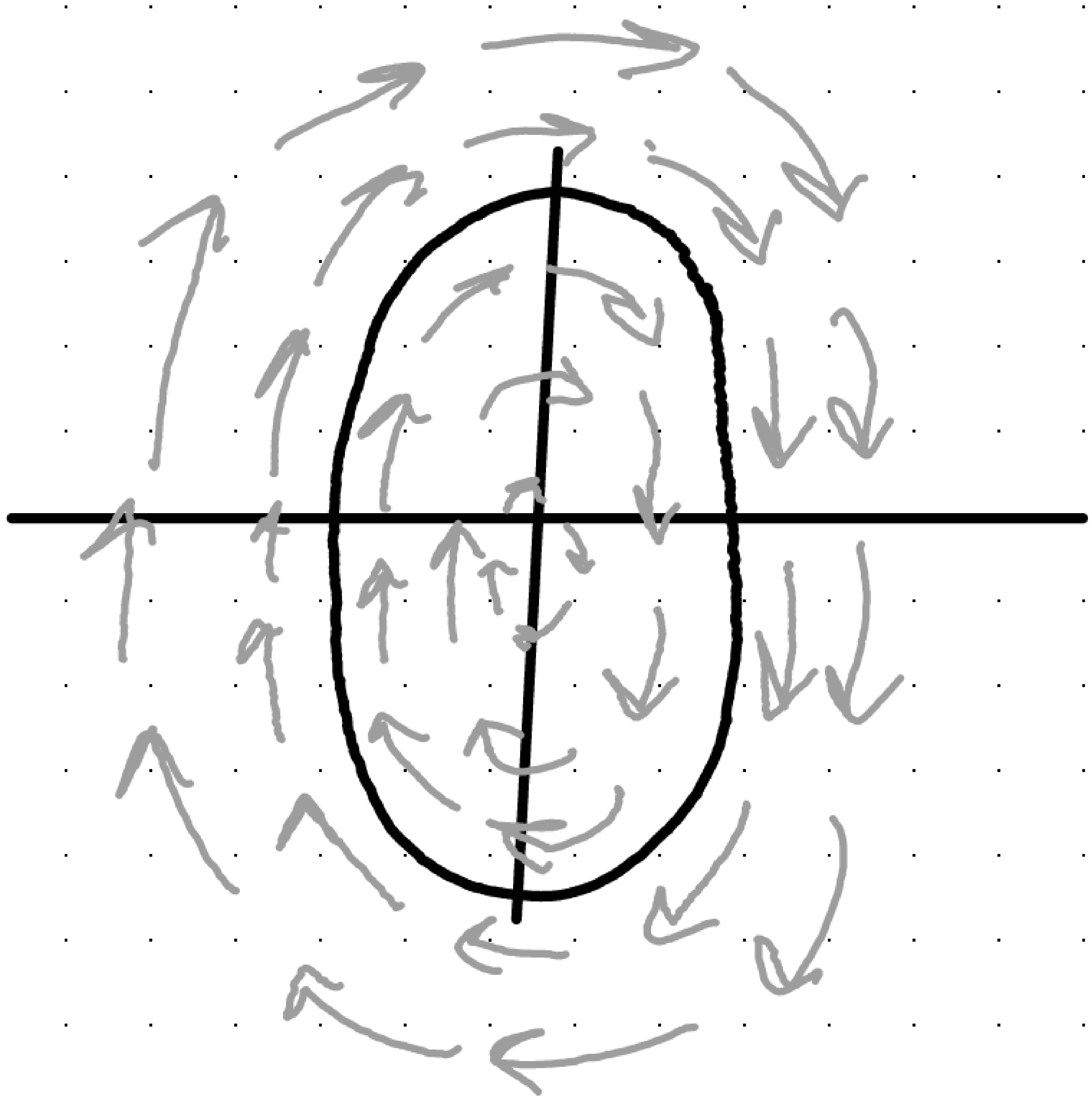
Example:

Consider the system

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \vec{x}$$

Two Eigenpairs are $2i \begin{bmatrix} 1 \\ 2i \end{bmatrix}$ & $-2i \begin{bmatrix} 1 \\ -2i \end{bmatrix}$

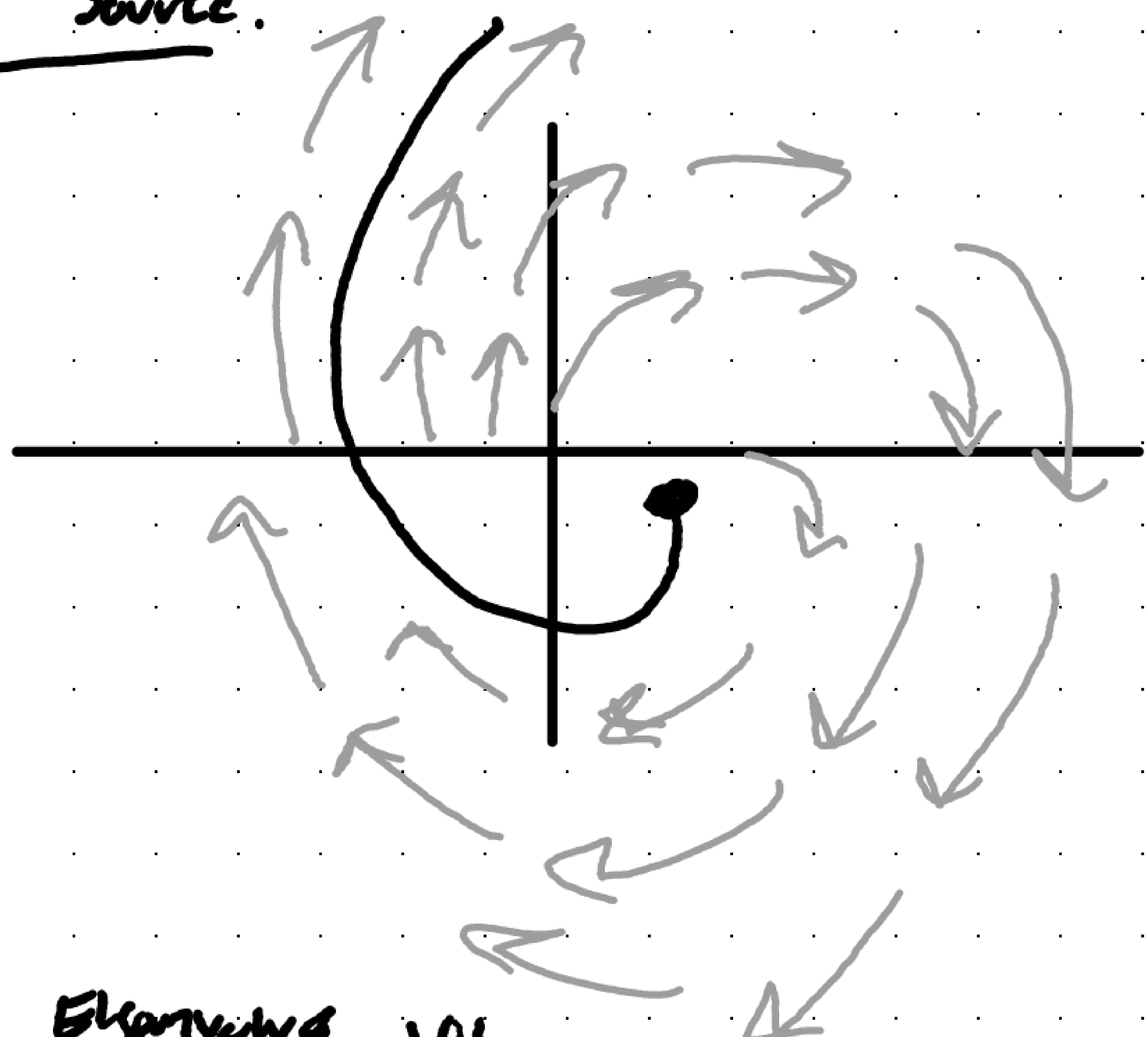
$\dot{x} = y$
$\dot{y} = -4x$



Center:

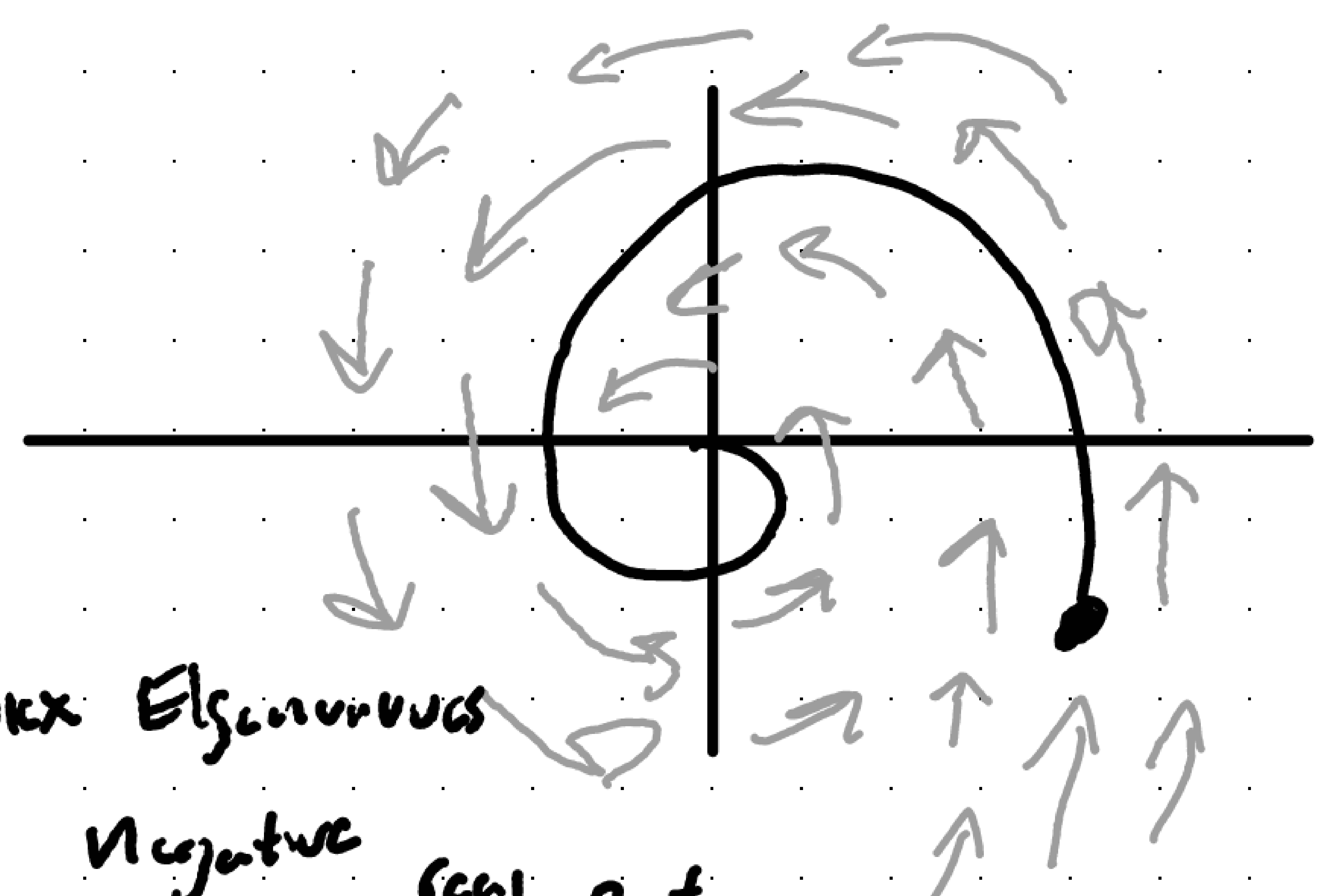
Eigenvalues are complex, with zero real part.

Spiral Source:



Complex Eigenvalues with positive real part,

Spiral Sink:



Complex Eigenvalues with negative real part.