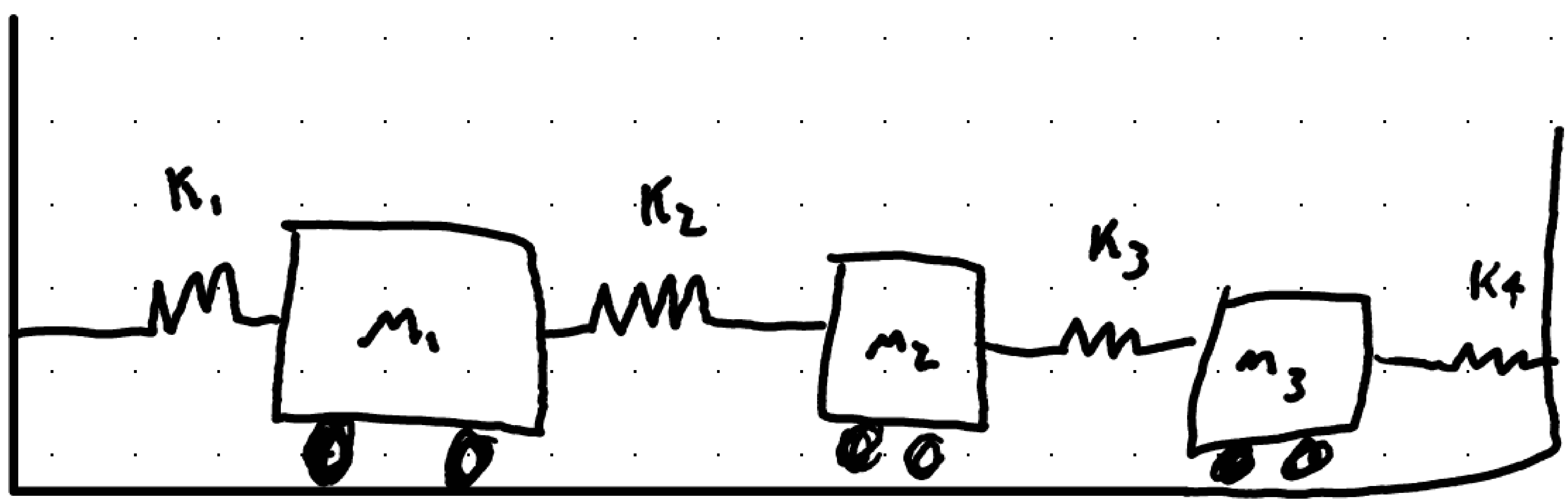
Undamped Spring Mass Systems

Let's Consider again Several Masses & Springs. For example, suppose m_1, m_2, m_3 are masses connected in series by four springs with spring constants K_1, K_2, K_3, K_4, sliding on a frictionless plane.
Let $x_i(t)$ denote the displacement from equilibrium of m_i



$$m_1 \ddot{x}_1 = -K_1 x_1 + K_2(x_2 - x_1) = -(K_1 + K_2)x_1 + K_2 x_2$$
$$m_2 \ddot{x}_2 = -K_2(x_2 - x_1) + K_3(x_3 - x_2) = K_1 x_1 - (K_2 + K_3)x_3 + K_3 x_3$$
$$m_3 \ddot{x}_3 = -K_3(x_3 - x_2) - K_4 x_3 = K_3 x_2 - (K_3 + K_4)x_3$$

$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}}_{m} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}'' = \underbrace{\begin{bmatrix} -(K_1 + K_2) & K_2 & 0 \\ K_1 & -(K_2 + K_3) & K_3 \\ 0 & K_3 & -(K_3 + K_4) \end{bmatrix}}_{K} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$m \vec{x}'' = K\vec{x}$$
$$\vec{x}'' = m^{-1} K \vec{x} = A\vec{x}$$

We look for solutions of the form

$$\vec{x}(t) = e^{\alpha t}\vec{v}$$

$$\vec{x}''(t) = \alpha^2 e^{\alpha t}\vec{v}$$

Plug this into $\vec{x}'' = A\vec{x}$:

$$\alpha^2 \cancel{e^{\alpha t}}\vec{v} = \cancel{e^{\alpha t}} A\vec{v}$$

$$A\vec{v} = \alpha^2 \vec{v}$$

α^2 is eigenvalue of $A = M^{-1}K$ with corresponding eigenvector $\vec{v}$

For our examples, the eigenvalues of $A = M^{-1}K$ will either be real & negative, or <u>zero</u>

If real & negative $\lambda = -\omega^2$, If $\alpha^2 = -\omega^2$, $\boxed{\alpha = \pm i\omega}$

$$\vec{x}(t) = e^{i\omega t}\vec{v} = (\cos\omega t + i\sin\omega t)\vec{v}$$

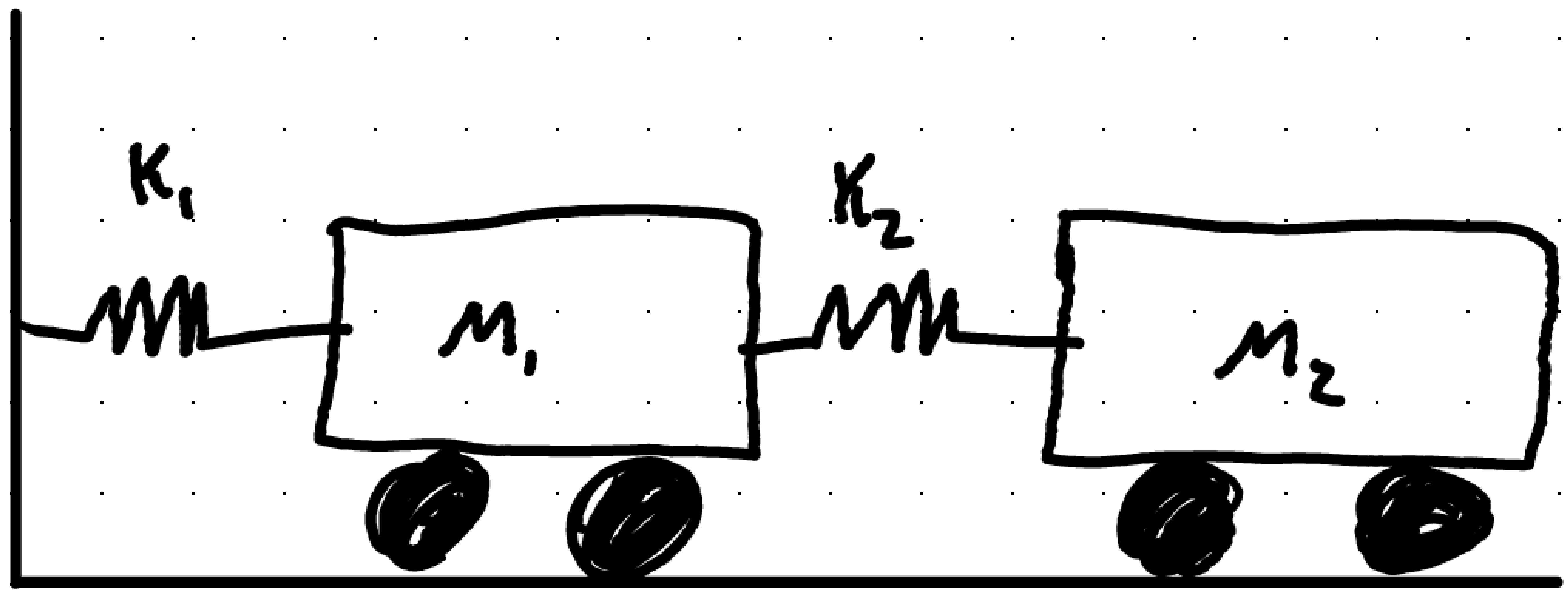
split real and imaginary

$\boxed{\vec{x}_1 = (a_1 \cos\omega t)\vec{v}}$, $\boxed{\vec{x}_2 = (b_1 \cos\omega t)\vec{v}}$

Ideally, we want a solution for each equation & each order.

If 0 is an eigenvalue with corresponding eigenvector $\vec{v}$, then $X(t) = \vec{v}(a+bt)$, for any a,b, is a solution.

Example. Consider the setup in the figure below, with $m_1 = 2\,kg$, $m_2 = 1\,kg$ $K_1 = 4\,N/m$ & $K_2 = 2\,N/m$



$$m_1 x_1'' = -K_1 x_1 + K_2(x_2 - x_1) = -(K_1 + K_2)x_1 + K_2 x_2$$
$$m_2 x_2'' = -K_2(x_2 - x_1) = K_2 x_1 - K_2 x_2$$

$$\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}}_{M}\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -(4+2) & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -6 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$M^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$, so $A = M^{-1}K = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -6 & 2 \\ 2 & -2 \end{bmatrix}$

$$= \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} -3-\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix}$$

$$= \lambda^2 + 5\lambda + 4 = (\lambda+1)(\lambda+4)$$

$$\lambda_1 = -1, \quad \lambda_2 = -4$$

...

$$\lambda = -1, \quad v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda = -4, \quad v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

From earlier

$\alpha^2 = \lambda$

$\alpha^2 = -1, \quad \alpha = \pm i$

$\omega_1 = 1$

$\alpha^2 = -4, \quad \alpha = \pm 2i$

$\omega_2 = 2$

So, general solution:

$$\vec{x}(t) = a_1 \cos(\omega_1 t)\vec{v}_1 + b_1 \sin(\omega_1 t)\vec{v}_1$$
$$+ a_2 \cos(\omega_2 t)\vec{v}_2 + b_2 \sin(\omega_2 t)\vec{v}_2$$

Two solutions for each eigenvalue!

Fill in the blanks:

$$a_1\cos t\begin{bmatrix}1\\2\end{bmatrix}+b_1\sin t\begin{bmatrix}1\\2\end{bmatrix}+a_2\cos 2t\begin{bmatrix}-1\\1\end{bmatrix}+b_2\sin 2t\begin{bmatrix}-1\\1\end{bmatrix}$$

Determine a_1, b_1, a_2, b_2 from IC's!

$$X(0)=a_1\vec{v}_1+a_2\vec{v}_2$$
$$\dot{X}(0)=\omega_1 b_1\vec{v}_1+\omega_2 b_2\vec{v}_2$$

or.....

$$\underbrace{a_1\cos t}_{\text{Scalar}}\underbrace{\begin{bmatrix}1\\2\end{bmatrix}}_{\text{Vector}}+\underbrace{a_2\cos 2t}_{\text{Scalar}}\underbrace{\begin{bmatrix}-1\\1\end{bmatrix}}_{\text{Vector}}+b_1\sin t\begin{bmatrix}1\\2\end{bmatrix}+b_2\sin 2t\begin{bmatrix}-1\\1\end{bmatrix}$$

So...

$$\underbrace{\begin{bmatrix}| & | \\ \vec{v}_1 & \vec{v}_2 \\ | & |\end{bmatrix}}_{E}\underbrace{\begin{bmatrix}\cos\omega_1 t & 0\\ 0 & \cos\omega_2 t\end{bmatrix}}_{C(t)}\begin{bmatrix}a_1\\a_2\end{bmatrix}+$$

$$\underbrace{\begin{bmatrix}| & | \\ \vec{v}_1 & \vec{v}_2 \\ | & |\end{bmatrix}}_{E}\underbrace{\begin{bmatrix}\sin\omega_1 t & 0\\ 0 & \sin\omega_2 t\end{bmatrix}}_{S(t)}\begin{bmatrix}b_1\\b_2\end{bmatrix}$$

and what we'll find is...

$$\vec{x}(0) = E\,\underbrace{c(0)}_{\hookrightarrow 1}\,\vec{a} + E\,\underbrace{s(0)}_{\hookrightarrow 0}\,\vec{b} = E\vec{a}$$

so..

$$\boxed{\vec{a} = E^{-1}\vec{x}(0)}$$

and

$$\vec{x}' = E\,\underbrace{c'(0)}_{\hookrightarrow 0}\,\vec{a} + E\,\underbrace{s'(0)}_{\hookrightarrow 1}\,\vec{b} = E\,s'(0)\,\vec{b}$$

so..

$$\boxed{\vec{b} = [s'(0)]^{-1} E^{-1}\vec{x}'(0)}$$

All to say:

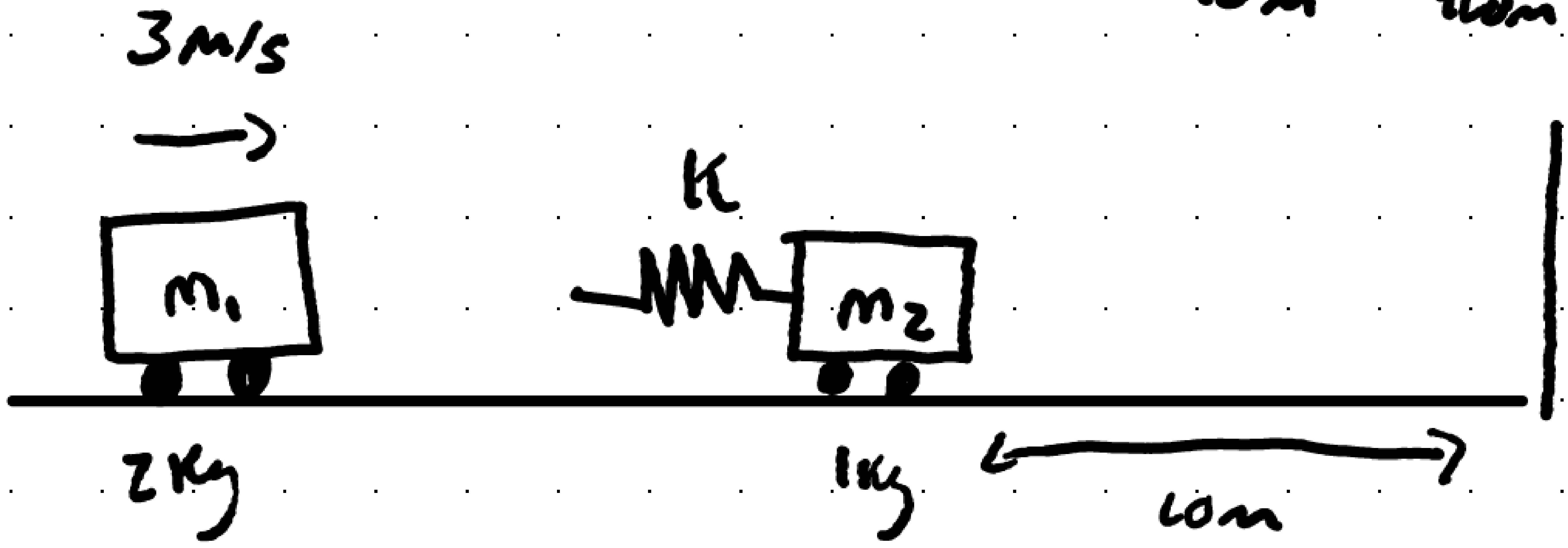
Given IC's

$$\boxed{\vec{x}(t) = E\,c(t)\,E^{-1}\vec{x}_0 + E\,s(t)\,[s'(0)]^{-1}E^{-1}\vec{x}_0'}$$

Can be useful if you are getting software to solve!

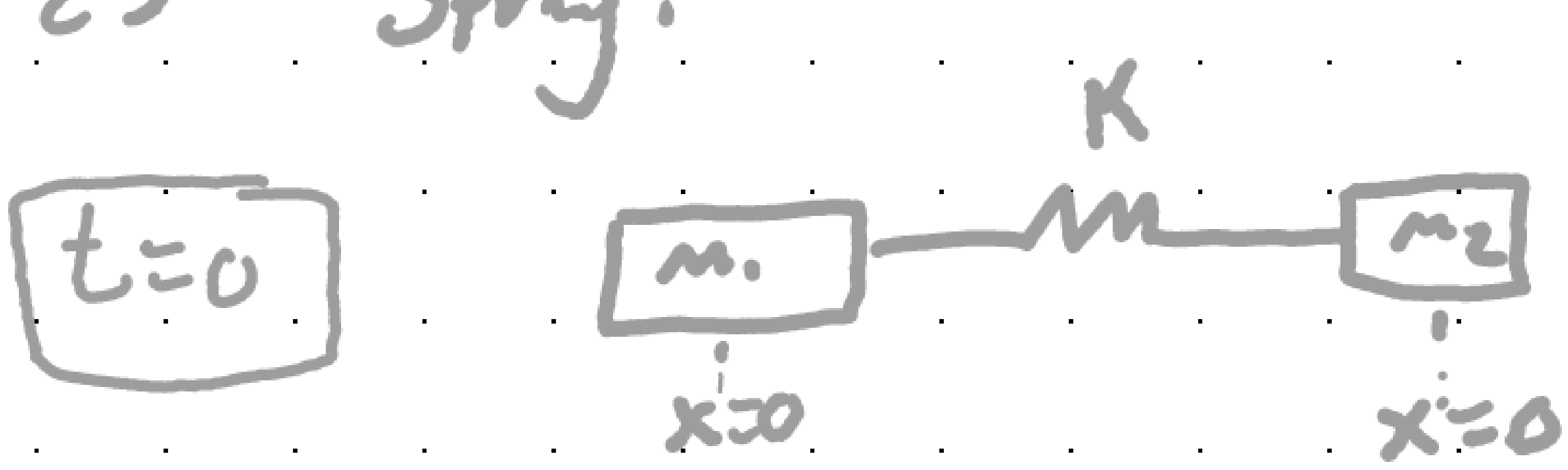
Example:

We have two toy rail cars. Car 1 of mass 2kg is traveling at 3m/s towards the second rail car that engages at the moment the cars hit (it connects to the cars) and does not let go. The bumper acts like a spring of spring constant $K = 2N/m$. The second car is 10m from the wall.



At what time after the cars link does impact with the wall happen? What is the speed of car 2 when it hits the wall?

Let $\boxed{t=0}$ correspond to moment car 1 strikes car 2's spring:



We can use our previous example!

Matrix Setup

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} -K & K \\ K & -K \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

$$A = M^{-1}K = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$$

0 is the eigenvalue of A, with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So is $\lambda = -3$, $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$\alpha^2 = -3$$
$$\alpha = \pm\sqrt{3}\, i$$

So $\vec{v} e^{\sqrt{3} i t}$ is solution

General Solution:

$$\vec{x}(t) = \vec{v}_1 (a_1 + b_1 t) + \vec{v}_2 (a_2 \cos\sqrt{3}t + b_2 \sin\sqrt{3}t)$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}(a_1 + b_1 t) + \begin{bmatrix} -1 \\ 2 \end{bmatrix}(a_2 \cos\sqrt{3}t + b_2 \sin\sqrt{3}t)$$

$$= \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \cos\sqrt{3}t \end{bmatrix}}_{C(t)} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} t & 0 \\ 0 & \sin\sqrt{3}t \end{bmatrix}}_{S(t)} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\vec{x}(0) = \vec{x}_0, \quad \vec{x}'(0) = \vec{x}_0'$$

$$\vec{x}(0) = E\vec{a} \Rightarrow \vec{a} = E^{-1}\vec{x}_0$$

$$\vec{x}'(0) = E(S'(0))\vec{b} \Rightarrow \vec{b} = [S'(0)]^{-1}E^{-1}x_0'$$

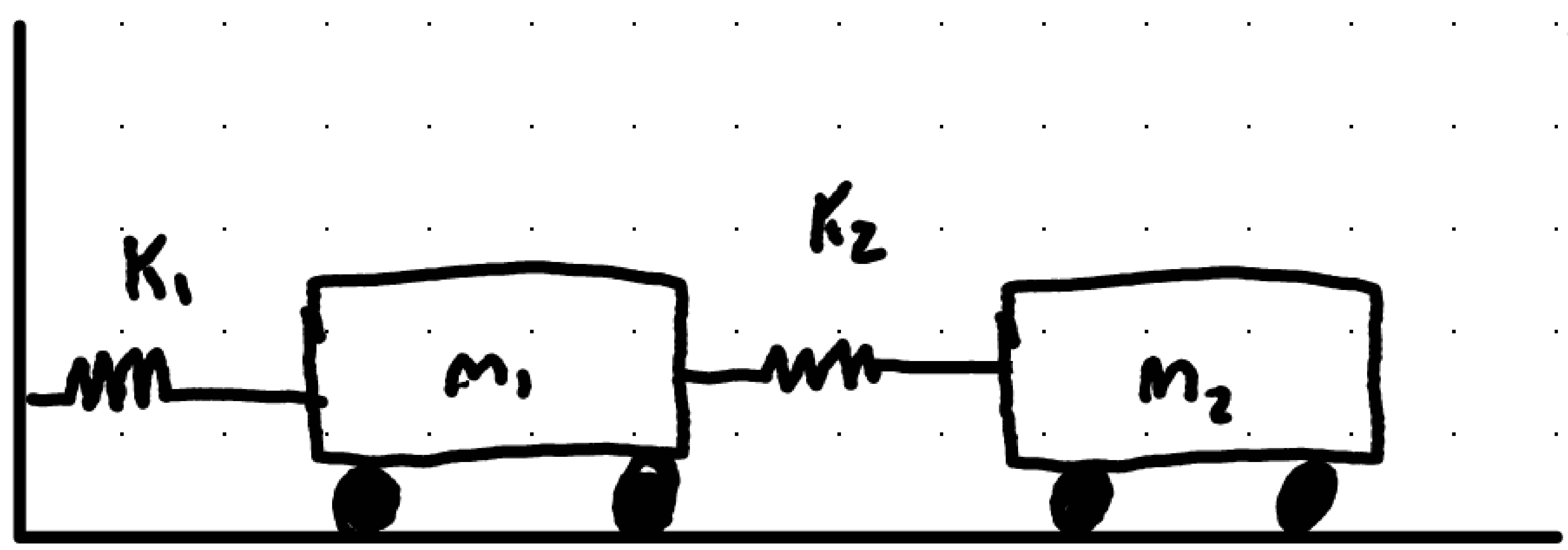
$$\vec{x}(t) = E\,C(t)\,E^{-1}\vec{x}_0 + E\,S(t)[S'(0)]^{-1}E^{-1}\vec{x}_0'$$

$$\vec{x}(0) = \vec{x}_0 = \begin{bmatrix}0\\0\end{bmatrix}, \quad \vec{x}'(0) = \vec{x}_0' = \begin{bmatrix}3\\0\end{bmatrix}$$

Adding Oscillator Forcings To Second Order DE Systems

Example. Consider again the setup in the figure below, with $m_1 = 2\,kg$ & $m_2 = 1\,kg$, $k_1 = 4\,N/m$, $k_2 = 2\,N/m$, but suppose a force of $\underline{2\cos 3t}$ (not pictured) acts on the second cart.

$\omega = 3$



$$x'' = A\vec{x} + F\cos 3t,$$

Forcing only on second cart

$$\vec{F} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Recall:

$$A = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \quad \omega_1 = 1, \quad \omega_2 = 2$$

So $\quad \vec{x}_p = \vec{C}\cos 3t$

$$\vec{c} = (A + \omega^2 I)^{-1}(-\vec{F})$$

$$A + \omega^2 I = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 2 & 7 \end{bmatrix}$$

$$(A + \omega^2 I)^{-1} = \frac{1}{40}\begin{bmatrix} 7 & -1 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} \frac{7}{40} & -\frac{1}{40} \\ -\frac{1}{20} & \frac{3}{20} \end{bmatrix}$$

Det

$$(A + \omega^2 I)^{-1}(-\vec{F}) = \begin{bmatrix} \frac{7}{40} & \frac{1}{40} \\ -\frac{1}{20} & \frac{3}{20} \end{bmatrix}\begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{20} \\ -\frac{3}{10} \end{bmatrix}$$

So, $\vec{x}_p = \begin{bmatrix} \frac{1}{20} \\ -\frac{3}{10} \end{bmatrix} \cos 3t$

$$x = \vec{x}_c + \vec{x}_p$$

Already found.

Remember that

$$\vec{x}_c = E\,C(t)\,\vec{a} + E\,S(t)\,\vec{b}$$

More on Next page

Remember that:

$$\vec{x}_c = E\,C(t)\,\vec{a} + E\,S(t)\,\vec{b}$$

&

$$E = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}, \qquad C(t) = \begin{bmatrix} \cos\omega_1 t & 0 \\ 0 & \cos\omega_2 t \end{bmatrix}$$

(Eigenvector matrix)

$$S(t) = \begin{bmatrix} \sin\omega_1 t & 0 \\ 0 & \sin\omega_2 t \end{bmatrix}$$

So the general solution is

$$\vec{x}(t) = \underbrace{E\,C(t)\,\vec{a} + E\,S(t)\,\vec{b}}_{\vec{x}_c} + \underbrace{\vec{c}\cos\omega t}_{\vec{x}_p}$$

$$\vec{x}(t) = \underbrace{E\,C(t)\,\vec{a} + E\,S(t)\,\vec{b}}_{\vec{x}_c} + \underbrace{\vec{c}\cos\omega t}_{\vec{x}_p}$$

So let's Apply IC's

$\boxed{\text{IC's}}$ $x(0) = \vec{x}_0$, $x'(0) = x_0'$:

$$\vec{x}(0) = E\vec{a} + \vec{c} \implies \vec{a} = E^{-1}(x_0 - \vec{c})$$

$$x'(0) = E S'(0)\vec{b} \implies \vec{b} = E^{-1}[S'(0)]^{-1} x_0'$$

This is the only difference from before!!!