Matrix Exponentials

Recall the taylor series for the exponential function.

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots + \frac{t^n}{n!} + \dots$$

Define, given a $n \times n$ matrix A:

$$\boxed{e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots}$$

Same for At, or any A

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$$

We can differentiate e^{At} as well and get

$$\frac{d}{dt} e^{At} = 0 + A + \frac{2At}{2!} + \frac{3(At)^2}{3!} + \dots$$

$$= A\left(I + At + \frac{(At)^2}{2!} + \dots\right)$$

$$= Ae^{At}$$

sub e back in!

This all leads to the following theorem.

Theorem: Let A be an $n \times n$ matrix. The general solution to $\vec{x}' = A\vec{x}$ is

$$\vec{x} = e^{tA}\vec{c}$$

where $\vec{c}$ is an arbitrary constant vector. In fact, $\vec{x}(0) = \vec{c}$.

Example. Compute e^A if $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

$$\left(A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},\ A^2 = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}, \dots,\ A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} a^3 & 0 \\ 0 & b^3 \end{bmatrix} + \dots$$

$$= \begin{bmatrix} 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \dots & 0 \\ 0 & 1 + b + \frac{b^2}{2!} + \frac{b^3}{3!} + \dots \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}}$$

And this is true for any diagonal matrix!

Example.

Compute e^A if $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

"Recall" Diagonalizability of matrices.

Fact:

If A is $n \times n$ (square), and has n L.I eigenvectors, then A is similar to a diagonal matrix:

$$E^{-1}AE = D$$

where D is a diagonal matrix with eigenvalues of A on the diagonal, and E is a matrix with columns of the eigenvectors.

For this Example: $E = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

So, $E^{-1}AE = D$, or $AE = ED$, or $A = EDE^{-1}$

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ ✓ Checks out!

Then

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$$

Cont. 2

Let's plug some stuff in..

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

$$= EE^{-1} + EDE^{-1} + \frac{(EDE^{-1})^2}{2!} + \frac{(EDE^{-1})^3}{3!} + \dots$$

$$= EE^{-1} + EDE^{-1} + \frac{ED\overset{I}{E^{-1}E}DE^{-1}}{2!} + \frac{ED\overset{I}{E^{-1}E}D\overset{I}{E^{-1}E}DE^{-1}}{3!} + \dots$$

$$= EE^{-1} + EDE^{-1} + \frac{ED^2E^{-1}}{2!} + \frac{ED^3E^{-1}}{3!} + \dots$$

$$= E\left(I + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \dots\right)E^{-1}$$

$$\boxed{= Ee^DE^{-1}}$$

And this is all we really need to do! Find E, and D!

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^3 & 0 \\ 0 & e^{-1} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} =$$

$$E e^{D} E^{-1} = \begin{bmatrix} \frac{1}{2}\left(\frac{1}{e}+e^{3}\right) & \frac{1}{2}\left(\frac{-1}{e}+e^{3}\right) \\ \frac{1}{2}\left(\frac{-1}{e}+e^{3}\right) & \frac{1}{2}\left(\frac{1}{e}+e^{3}\right) \end{bmatrix}$$

A Reminder of Fundamental Matrix Solutions:

Columns are L.I Solutions.

Say $e^{t}\begin{bmatrix}1\\1\end{bmatrix}$, $e^{4t}\begin{bmatrix}1\\2\end{bmatrix}$, Solve a system,

then $x(t) = \begin{bmatrix}e^{t} & e^{4t}\\ e^{t} & 2e^{4t}\end{bmatrix}$ is a fundamental Matrix Solution.

i.e. $x'(t) = A\,x(t)$

Note: $x(0) = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix} = E$

Eigenvector matrix

Example:

Use Matrix Exponentials to Solve $\vec{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x}$, with the IC, $\vec{x}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

Matrix Exponentials

$$e^{A} = \frac{1}{2}\begin{bmatrix} e^{3} + e^{-1} & e^{3} - e^{-1} \\ e^{3} - e^{-1} & e^{3} + e^{-1} \end{bmatrix},$$

$$e^{At} = \frac{1}{2}\begin{bmatrix} e^{3t} + e^{-t} & e^{3t} - e^{-t} \\ e^{3t} - e^{-t} & e^{3t} - e^{-t} \end{bmatrix}$$

$\vec{x}(t) = e^{At}\vec{c}$ ← From Earlier

$$\vec{x}(t) = \frac{1}{2}\begin{bmatrix} e^{3t} + e^{-t} & e^{3t} - e^{-t} \\ e^{3t} - e^{-t} & e^{3t} - e^{-t} \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3e^{3t} + e^{-t} \\ 3e^{3t} - e^{-t} \end{bmatrix}$$

While now, you have to solve for the matrix exponential, plugging in the initial conditions is Super Easy!