Non-homogeneous Systems

We'll briefly explore a couple of ways to solve non-homogeneous systems.

Integrating Factor / Variation of Parameters Method:

If $X(t)$ is a fundamental matrix solution to $\vec{x}' = Ax$, then:

$$\vec{x}_p = X(t) \int [X(t)]^{-1} f(t)\, dt$$

is a particular solution to $\vec{x}' = A\vec{x} + \vec{f}(t)$

Undetermined Coefficients Method:

Look for a particular solution based on the form of the non-homogeneous term.

Example: Find the general solution of the system:

$$\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} e^t \\ t \end{bmatrix}$$

Homogeneous Solution: $\vec{x}_c = Ce^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Forcing: $\vec{f}(t)$

$$\underbrace{\vec{f}(t) = \begin{bmatrix} e^t \\ t \end{bmatrix} = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{Here, I would extract the forcings for simplicity}}$$

Okay, let's make a guess now.

Try:

$$\vec{x}_p = e^t \vec{a} + t\vec{b} + \vec{c}$$

Warning! There is a Dupe!

So for systems, we add another term.

$$\boxed{\vec{x}_p = te^t \vec{a} + e^t \vec{b} + t\vec{c} + \vec{d}}$$

$$\vec{x}_p = te^t\vec{a} + e^t\vec{b} + t\vec{c} + \vec{d}$$

$$\vec{x}_p{}' = te^t\vec{a} + e^t\vec{a} + e^t\vec{b} + \vec{c}$$
$$= te^t\vec{a} + e^t(\vec{a} + \vec{b}) + \vec{c}$$

$$\vec{x}_p{}' = A\vec{x}_p$$

$$te^t\vec{a} + e^t(\vec{a} + \vec{b}) + \vec{c} = te^tA\vec{a} + e^tA\vec{b} + tA\vec{c} + A\vec{d} + e^t\begin{bmatrix}1\\0\end{bmatrix} + t\begin{bmatrix}0\\1\end{bmatrix}$$

So,

$$\cancel{te^t}\vec{a} = \cancel{te^t}A\vec{a}$$

$$\cancel{e^t}(\vec{a}+\vec{b}) = \cancel{e^t}(A\vec{b} + \begin{bmatrix}1\\0\end{bmatrix}$$

$$A\vec{c} + \begin{bmatrix}0\\1\end{bmatrix} = 0$$

$$A\vec{d} = \vec{c}$$

Systems of Equations

$A\vec{a} = \vec{a}$

$2a_1 - a_2 = a_1$
$3a_1 - 2a_2 = a_2$

$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\vec{a} = \vec{a}$

$A\vec{b} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{a} + \vec{b}$

$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\vec{b} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{a} + \vec{b}$

$2b_1 - b_2 + 1 = a_1 + b_1$
$3b_1 - 2b_2 = a_2 + b_2$

$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\vec{c} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{0}$

$2c_1 - c_2 = 0$
$3c_1 - 2c_2 + 1 = 0$

$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\vec{d} = \vec{c}$

$2d_1 - d_2 = c_1$
$3d_1 - 2d_2 = c_2$

$a_1 = \frac{3}{2}, \quad a_2 = \frac{3}{2}$

$b_1 = \frac{-1}{4}, \quad b_2 = \frac{-3}{4}$

$c_1 = 1, \quad c_2 = 2$

$d_1 = 0, \quad d_2 = -1$

So,

$$x_p(t) = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix} te^t + \begin{bmatrix} -1/4 \\ -3/4 \end{bmatrix} e^t + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} te^t - \frac{1}{4} e^t + t \\ \frac{3}{2} te^t - \frac{3}{4} e^t + 2t - 1 \end{bmatrix}$$

Either Method Can be Used, but for Certain problems, One Can be Vastly more efficient than the Other. In this Case, MVC Proved to be Waaaay Slower than IF/VOP Method. One Should not discredit MVC Method based on just this example.