Review

Overdamped $c^2 > 4mk$ $\lambda =$ 2 real distinct root

$$y_h = Ae^{\lambda t} + Be^{\lambda t}$$

Critically Damped

$c^2 = 4mk$ $\lambda =$ Single real repeated root

$$y_h = Ae^{\lambda t} + Be^{\lambda t}$$

Underdamped

$c^2 < 4mk$ $\lambda = a \pm ib$

$$y_h = e^{\alpha} (A\cos\beta t + B\sin\beta t)$$

Comes from Eulers formula

$$Ae^{(a+ib)t} + Be^{(a-ib)t}$$

Example

$$\ddot{y} + 2\dot{y} + y = e^{-t}$$

$$\lambda^2 + 2\lambda + 1 = 0 \qquad \lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$= \frac{-(2) \pm \sqrt{2^2 - 4(1)(1)}}{2(1)}$$

$c^2 = 4mk$ Critically Damped

$\lambda = -1$

$$y_h = C_1 e^{-1t} + C_2 t e^{-1t}$$

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y_p: e^{-t}

Guess Ae^{-t} dupe!

$\downarrow$

tAe^{-t} dup!

$\downarrow$

t^2Ae^{-t} good.

$$y_p = At^2e^{-t}$$

THIS IS A PRODUCT!!!

$$y'_p = A(2te^{-t} + t^2(-e^{-t}))$$

$$= Ae^{-t}(2t - t^2) \therefore$$

$$y''_p = A(-e^{-t}(2t - t^2) + e^{-t}(2 - 2t))$$

$$= Ae^{-t}((-2t + t^2) + (2 - 2t)$$

$$= Ae^{-t}(t^2 - 4t + 2) \therefore$$

$$\ddot{y} + 2\dot{y} + y = e^{-t}$$

$$Ae^{-t}(t^2 - 4t + 2) + 2(Ae^{-t}(2t - t^2)) + At^2e^{-t} = e^{-t}$$

$$Ae^{-t}(t^2 - 4t + 2) + 2(2t - t^2) + t^2) = e^{-t}$$

t^2: $t^2 - 2t^2 + t^2 = 0$

t: $-4t + 4t = 0$

Const: $2 = e^{-t}$

$$A\cancel{e^{-t}}(2) = \cancel{e^{-t}}$$

$$\boxed{A = \frac{1}{2}}$$

$$\boxed{y_p = \frac{1}{2} t^2 e^{-t}}$$

$y = y_h + y_p$

$$\boxed{y = c_1 e^{-1t} + c_2 t e^{-1t} + \frac{1}{2} t^2 e^{-t}}$$

Now, solve initial conditions, $y(0) = 0$
$y'(0) = 1$

$y(0) = c_1(1) + 0 + 0$

$$\boxed{c_1 = 0}$$

$\dot{y} = -e^{-t}(c_2 t + \frac{1}{2}t^2) + e^{-t}(c_2 + t)$

$$\boxed{c_2 = 0}$$

$$\boxed{y(t) = e^{-t}(t + \frac{1}{2}t^2)}$$

Final Answer!

Example 2:

$$\boxed{\ddot{y} + 2y' + 5y = e^{-t}\cos(2t)}$$

y_h

$$e^{\lambda t}(\lambda^2 + 2\lambda + 5) = 0$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$m=1$ $c=2$ $k=5$

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$\frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$\lambda \Rightarrow \frac{-2 \pm 4i}{2} \Rightarrow -1 \pm 2i$$

Underdamped

$$\sqrt{-16}$$

$$i\sqrt{16} \to 4i$$

$$y_h = e^{at}(c_1 \cos bt + c_2 \sin bt)$$

$$y_h = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$$

y_p $\boxed{\ddot{y} + 2y' + 5y = e^{-t}\cos(2t)}$

$y_h = e^{-t}(C_1\cos 2t + C_2\sin 2t)$

Duplications

Compare

$e^{-t}(C_1\cos 2t + C_2\sin 2t) \leftrightarrow e^{-t}\cos 2t$

① Check $\lambda = a \pm ib$

a and b match

Dupe. ☆

OR

Use eulers to convert $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{i2t} = \cos 2t + i\sin 2t$

$\cos 2t = RE[e^{i2t}]$

$$\ddot{y} + 2y' + 5y = e^{-t}RE[e^{i2t}]$$

$$\downarrow$$

$$\ddot{y} + 2\dot{y} + 5y = RE[e^{-t}e^{i2t}]$$

$$\downarrow$$

$$\ddot{y} + 2\dot{y} + 5y = RE[e^{(-1+2i)t}]$$

We have to factor out t!!

AND we see λ here as well! We can check for Dupe here as well Dupe.

Aux problem

$$\ddot{Y} + 2\dot{Y} + 5Y = e^{(-1+2i)t}$$

$\dots\ y_p = Re[Y_p]$

Aux problem Cont....

Let's deal with the dupe now!

$$\ddot{Y} + 2\dot{Y} + 5Y = e^{(-1+2i)t}$$

Lets make our guesses!

Dupe Dealt with!

$$Y_p = Ate^{(-1+2i)t}$$

* Remember, this is a product.....

$$\dot{Y}_p = A\left(e^{(-1+2i)t} + t(-1+2i)e^{(-1+2i)t}\right)$$

↳ $Ae^{(-1+2i)t}\left(1 + t(-1+2i)\right)$

* Remember, this is a product.....

$$\ddot{Y}_p = A\left((-1+2i)e^{(-1+2i)t}[1+t(-1+2i)] + e^{(-1+2i)t}(-1+2i)\right)$$

↳ $Ae^{(-1+2i)t}\left((-1+2i)(1+2i)t + 1)\right)$

↳ $Ae^{(-1+2i)t}\left(2+(-1+2i)t\right)$

Let's Sub in!

$$\underbrace{Ae^{(-1+2i)t}(2+(-1+2i)t)}_{\ddot{Y}} + 2\underbrace{\left(Ae^{(-1+2i)t}(1 + t(-1+2i))\right)}_{\dot{Y}} + 5\underbrace{\left(-te^{(-1+2i)t}\right)}_{Y} = e^{(-1+2i)t}$$

Let's Clean that up a bit....

$$Ae^{(-1+2i)t}\Big((-1+2i)(2+(-1+2i)t)+2(1+(-1+2i)t)+5t$$

$$= e^{(-1+2i)t}$$

$$\downarrow$$

A LOT OF SKIPPED

Simplification Steps

$$\downarrow$$

$$Ae^{(-1+2i)t}(-2+4i-3t-4i)+(2-2t+4it)+5t=e^{(-1+2i)t}$$

$$\hookrightarrow Ae^{(-1+2i)t}(4i)=e^{(-1+2i)t}$$

$$A(4i)=1$$

$$\boxed{A=\frac{1}{4i}}$$

$$Y_p=\frac{1}{4i}te^{(-1+2i)t}$$

don't forget about Me!

* We need to get rid of i in the denominator and i in the exponent

$$\rightarrow \quad \frac{1}{4i}\times\frac{i}{i}=\frac{-i}{4}$$

$$e^{-t+2it}=e^{-t}e^{2it}=e^{-t}\cos 2t+i\sin 2t$$

So...

$$Y_p = -\frac{i}{4} t e^{-t} (\cos 2t + i \sin 2t)$$

We have a problem here!
Be careful when expanding...

$$Y_p = \frac{-te^{-t}}{4} (i\cos 2t - \sin 2t)$$

$$\boxed{y_p = RE[Y_p] = \frac{te^{-t}}{4} \sin 2t}$$

$$y = \underbrace{e^{-t}(c_1\cos 2t + c_2\sin 2t)}_{y_h} + \underbrace{\frac{te^{-t}}{4}\sin 2t}_{y_p}$$