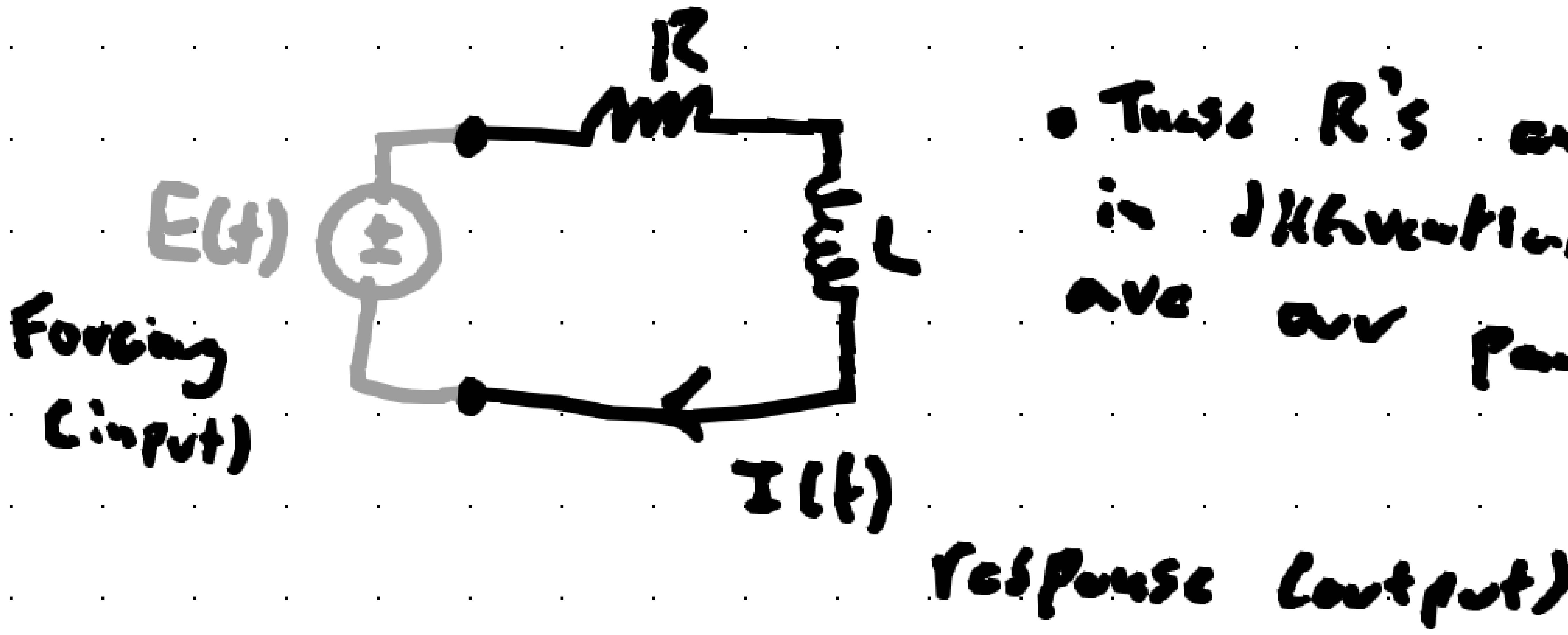
First order LTI Systems

Consider a Series RL Circuit with a voltage supply $E(t)$:



- These R's and L's in differential Equations are our parameters

In DE, we take sitting systems, and drive them with an input. We then get a corresponding output.

From KVL, we have the model

$$\underbrace{L\frac{dI}{dt} + RI}_{\substack{\text{LTI} \\ \text{"System"}}} = \underbrace{E(t)}_{\text{Forcing}}$$

Linear

model only involves linear terms in I

$I, \frac{dI}{dt}, \frac{dI^2}{d^2t}$

Time-**I**nvariant "constant coefficient"

Parameters are static in time

No matter if I power a circuit at 8am, or 8pm, it will be the same.

Assume DC Voltage Source

$E(t) = E_0$ and initial Condition

$I(0) = I_0$

$\dot{I} = \frac{dI}{dt}$

$L\dot{I} + RI = E_0, \quad I(0) = I_0$

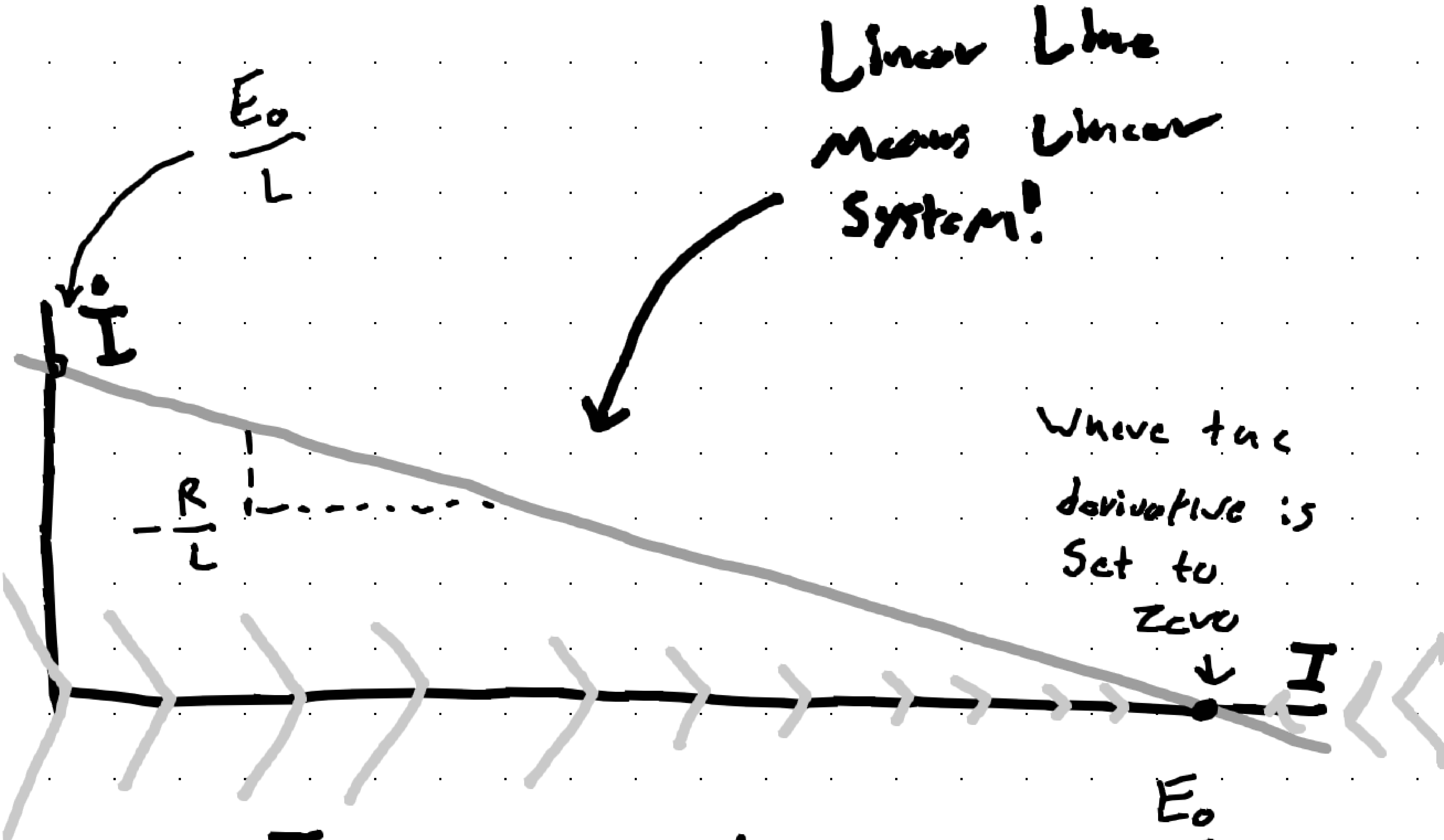
State Space

The state of the system at time t is given by $(I(t), \dot{I}(t))$

If we re-arrange our DE,

$$\dot{I}(t) = \frac{E_0}{L} - \frac{R}{L} I(t)$$

State Space Diagram



Increasing current!
(derivative is positive)

$I = \frac{E_0}{R}$ is a Stable Fixed point of the System.

↑

The zero point on the graph

(i) Fixed Point $\dot{I} = 0$ → No change in I.

→ system in Steady State

(ii) Stable: The System Is <u>attracted</u> to this fixed point

↳ This System can be <u>unstable</u> This means it repelled from the Point.

unstable.

Money is unstable or instable!

Now, let's Solve this dE!

we are Searching for some current that makes

$$L\dot{I} + RI = E_0$$

True!

(This is hard!)

Method of Undetermined Coefficients (MUD)

$$L\dot{I} + RI = E_0, \quad I(0) = I_0$$

1. Get homogenous Solution I_h (natural Response)

$$L\dot{I}_h + RI_h = 0$$

Zero forcing.

(What happens if we do nothing to the system?)

Exponentials are really good for first order LTI Systems!

They grow and decay exponentially!

$$\text{Assume } I_h = Ce^{\lambda t}$$

Free Coefficient

differentiating

$$\dot{I}_h = \lambda \underbrace{Ce^{\lambda t}}_{\text{Eigenfunction}}$$

Eigenvalue

We got the same thing! This is why this works!

Any multiple of the Eigenfunction is the same!

Plugging back in...

$$L(\lambda Ce^{\lambda t}) + R(Ce^{\lambda t}) = 0$$

$$Ce^{\lambda t}(L\lambda + R) = 0$$

This is the Characteristic Equation

$$L\lambda + R = 0$$

$$\lambda = -\frac{R}{L}$$

Stable

$$I_h = Ce^{-\frac{R}{L}t}$$

rate $[s^{-1}]$

This is a decaying Exponential

This because time is linear! You cannot have time in an exponential.

Is also equal to

$$I_h = Ce^{-t/\tau}$$

PTC ↓

Process Time Constant

$$\tau = \frac{L}{R}$$

[s]

* This characterizes the speed of the system's response.

$$L\dot{I} + RI = E_0$$

$$\tau \dot{I} + I = \frac{E_0}{R}$$

$$[s]\ [A/s]\ [A]\ \ [A]$$

2. Get Particular Solution

$$L\dot{I}_p + RI_p = E_0$$

(This is kind of the same, but without the $+0$ from the homogeneous)

Linear systems mimic their forcings.

Here we Have...

Constant forcing $E_0 \Rightarrow$ Constant Response $I_p = D$

Undetermined Coefficient.

So now we have...

$$L(0) + RD = E_0$$

$$D = \frac{E_0}{R}$$

$$I_p = \frac{E_0}{R}$$

... Hey... This looks like our fixed point on the graph...

3. Satisfy Initial Condition

$$I(0) = I_0$$

$$L\dot{I} + RI = E$$

$$L(\underline{\dot{I}_p} + \dot{I}_h) + R(\underline{I_p} + I_h) = \underline{E_0} + 0$$

- A mix of the particular, and homogeneous system!

$$I = I_h + I_p$$

$$I(t) = Ce^{-\frac{R}{L}t} + \frac{E_0}{R}$$

$$I(0) = Ce^{-\frac{R}{L}(0)} + \frac{E_0}{R}$$

I_0 1

$$C = I_0 - \frac{E_0}{R}$$

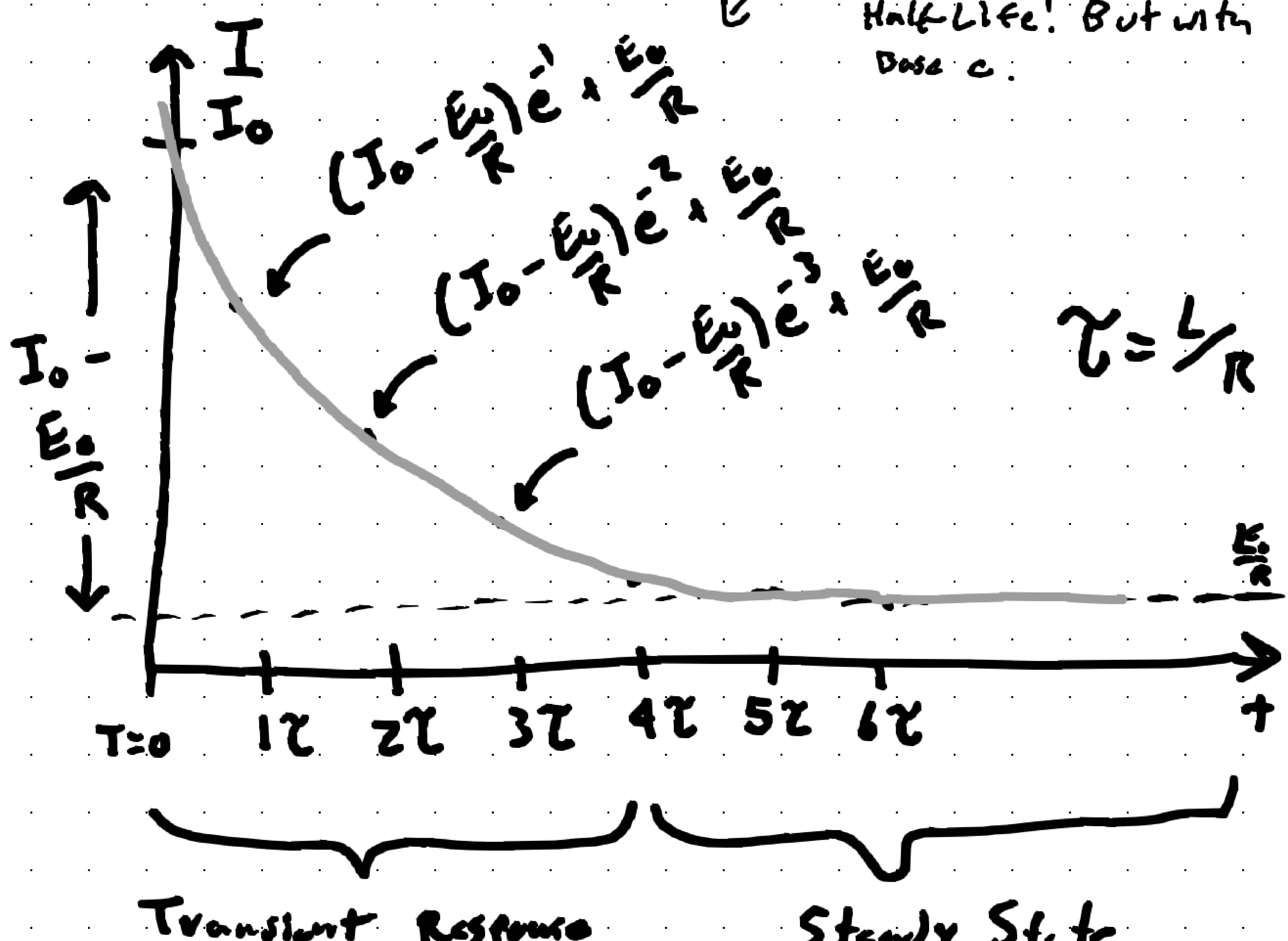
$$\boxed{I(t) = \left(I_0 - \frac{E_0}{R}\right)e^{-\frac{R}{L}t} + \frac{E_0}{R}}$$

4. Sketch $I(t)$

$$I(t) = \left(I_0 - \frac{E_0}{R}\right)e^{-\frac{R}{L}t} + \frac{E_0}{R}$$

$T \geq 3\tau$, $I(t) = I_p$, response in steady state.

This is known as e-folding. Very similar to Half-Life! But with base e.

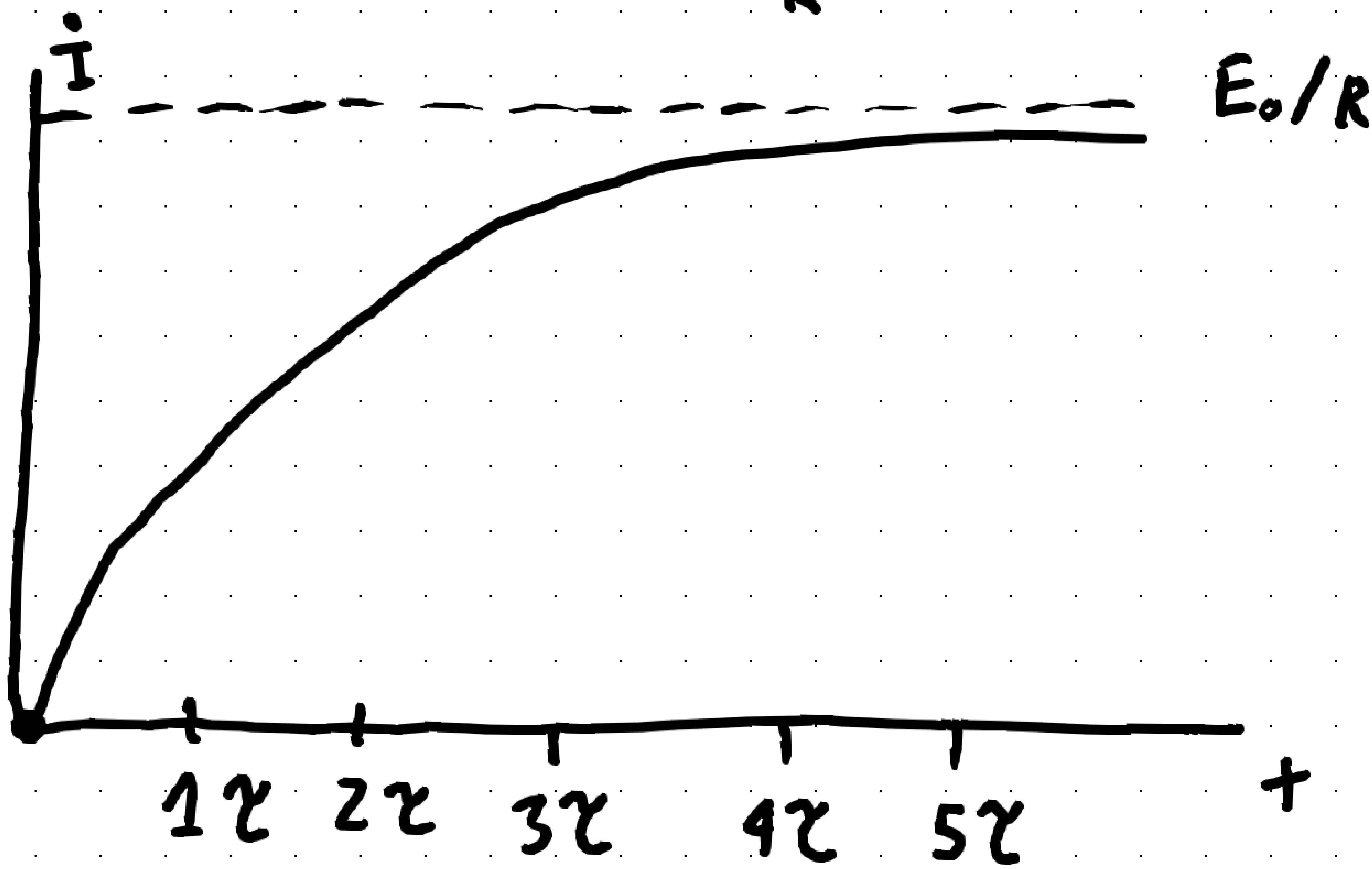


This is really just a decaying exponential! Shifted up.

$$I(t) = \underbrace{\left(I_0 - \frac{E_0}{R}\right)e^{-\frac{R}{L}t}}_{\text{Transient Response}} + \underbrace{\frac{E_0}{R}}_{\text{Steady Response}}$$

Past about 3τ, the system has no memory of its initial state.

$I_0 = 0 \qquad I(t) = \frac{E_0}{R}\left(1 - e^{-\frac{R}{L}t}\right)$



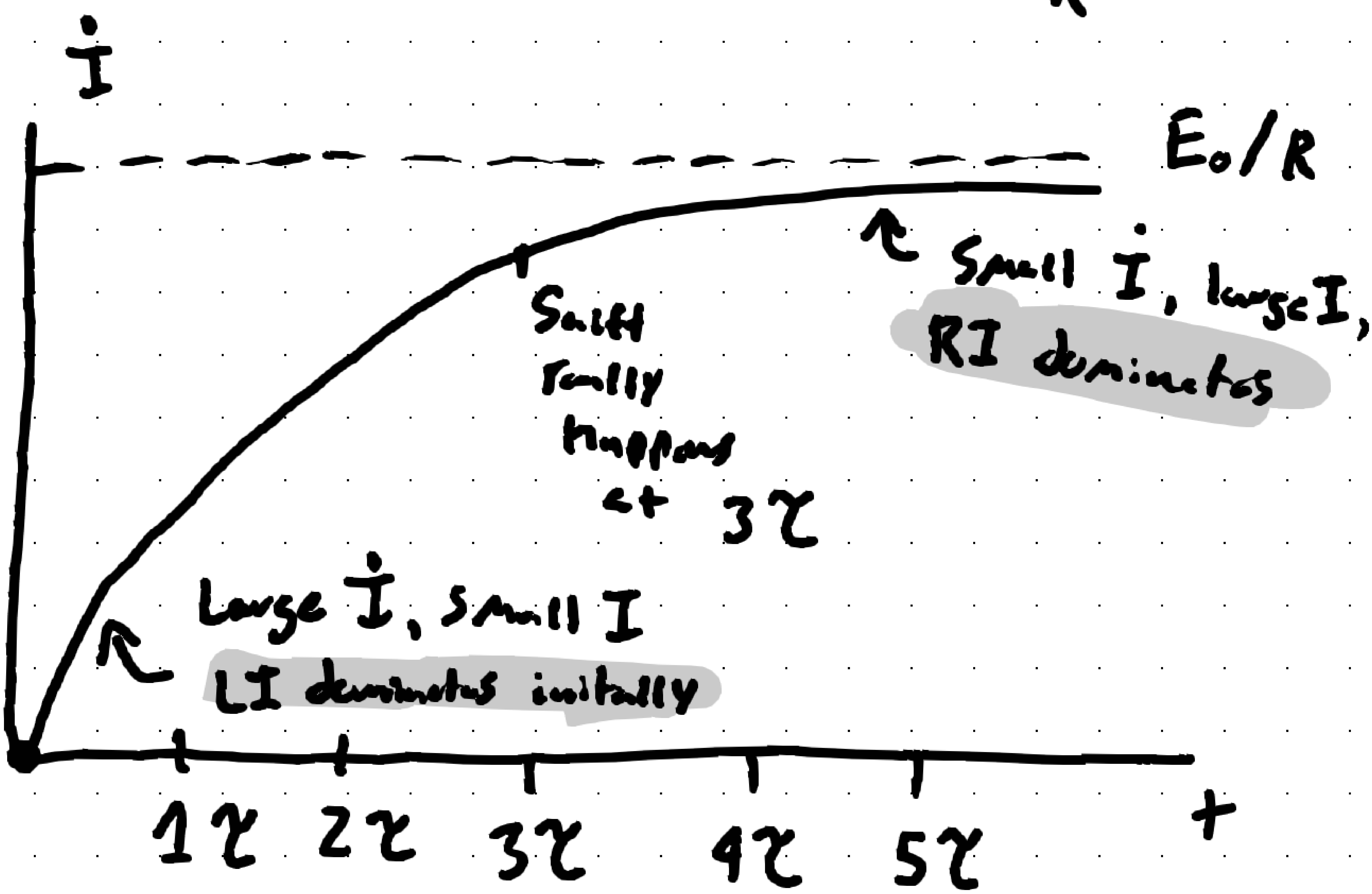
t	I(t)	
1τ	$0.63\, E_0/R$	Transient
2τ	$0.86\, E_0/R$	Transient
3τ	$0.95\, E_0/R$	Transient
4τ	$0.98\, E_0/R$	Steady State
5τ	$\sim E_0/R$	Steady State

$$\underbrace{L\dot{I}} + \underbrace{RI} = \underbrace{E_0}$$

Voltage drop across an inductor

Voltage drop across resistor.

Total voltage to allocate across L and R



So initally, the fed Voltage is being used by Mostly the inductor!

Where as past 3τ, or at a later time the resistor dominates.

At small time:

$e^{u} = 1 + u$

near $u = 0$

$$I(t) \approx \cancel{\frac{E_0}{R}}\left[\cancel{1} - (\cancel{1} - \frac{\cancel{R}}{L}t)\right]$$

$$\boxed{\approx \frac{E_0}{L}t \quad \text{linear approximation}}$$