Differential Equations — Lecture Five

$$\underbrace{\tau\dot{y} + y}_{\text{System}} = y_1\frac{t}{t_1} + y_2 + y_3 e^{-t/\alpha} + y_4\frac{t}{t_4}e^{-\beta t}$$

$y(0) = y_0 \qquad \alpha \neq \tau, \; \beta = \frac{1}{\tau}$

(1) Solve homogeneous y_h

$$\tau\dot{y}_h + y_h = 0$$

Steps skipped

$$\tau\lambda + 1 = 0$$

$$\lambda = -\frac{1}{\tau}$$

System stable

$$y_h = Ce^{-t/\tau}$$

② Get particular Solution y_p

$$\underbrace{\underbrace{y_1 \frac{t}{t_1}}_{\text{Linear}} + \underbrace{y_2}_{\text{Linear}}} + \underbrace{y_3 e^{-t/\alpha}}_{\text{Exponential}} + \underbrace{y_4 \frac{t}{t_4} e^{-\delta t}}_{\text{Exponential}}$$

$$y_p = y_{p_1} + y_{p_2} + y_{p_3}$$

① y_{p_1}

$$\tau \dot{y}_{p_1} + y_{p_1} = y_1 \frac{t}{t_1} + y_2$$

(Guess y_{p_1}, $At + B$ for linear)

$$\tau (A) + (\underline{At + B}) = y_1 \frac{t}{t_1} + y_2$$

$$At + B + A\tau = \frac{y_1}{t_1} t + y_2$$

Equating coefficients!

t^1: $A = \frac{y_1}{t_1}$

t^0: $B + A\tau = y_2$

↳ $B = y_2 - \frac{y_1}{t_1}\tau$

$$y_{p_1} = y_1 \frac{t-\tau}{t_1} + y_2$$

② y_{p_2}

$$\tau \dot{y}_{p_2} + y_{p_2} = y_3 e^{-t/\alpha}, \quad \alpha \neq \tau$$

$$y_{p_2} = D e^{-t/\alpha} \leftarrow$$ Exponential Guess *

$$\dot{y}_p = -\frac{1}{\alpha} D e^{-t/\alpha}$$

$$\tau\left(-\frac{1}{\alpha} D e^{-t/\alpha}\right) + D e^{-t/\alpha} = y_3 e^{-t/\alpha}$$

$$e^{-t/\alpha} D\left(-\frac{\tau}{\alpha} + 1\right) = y_3 e^{-t/\alpha}$$

$$D = \frac{y_3}{1 - \tau/\alpha}, \quad \alpha \neq \tau$$

$$y_{p_2} = y_3 \frac{1}{1 - \tau/\alpha} e^{-t/\alpha}$$

③ y_{p_3}

$$\tau \dot{y}_{p_3} + y_{p_3} = y_4 \frac{t}{t_4} e^{-\beta t}, \quad \beta = 1/\tau$$

$$y_{p_3} = (\underbrace{D_1 t + D_2}_{\text{Linear basis}}) \underbrace{e^{-\beta t}}_{\text{Exponential basis}}$$

$y_p = D_2 e^{-\beta t}$

$y_h = C e^{-t/\tau}$

However, This dupes y_h!!!...

So, multiply by t

$$y_{p_3} = (D_1 t^2 + D_2 t) e^{-\beta t}$$

* New Awesome basis

$$\dot{y}_{p_3} = e^{-\beta t}(-\beta(D_1 t^2 + D_2 t) + 2D_1 t + D_2)$$

And its derivative

$$\tau \dot{y}_{p_3} + y_{p_3} = e^{-\beta t}\left[(D_1 - \tau\beta D_1)t_2 + (D_2 - \tau\beta D_2 + 2\tau D_1)t + \tau D_2\right]$$

$$e^{-\beta t}\left[(D_1 - D_1)t^2 + (D_2 - D_2 + 2\tau D_1)t + \tau D_2\right] = \frac{y_4}{t_4} t$$

t^2: $D_1 - D_1 = 0$

t^1: $D_1 - D_2 + 2\tau D_1 = \frac{y_4}{T_4}$, $D_1 = \frac{y_4}{2\tau t_4}$

t^0: $\tau D_2 = 0 \Rightarrow D_1 = 0$

$$y_{p_3} = \left(D_1 t^2 + D_2\right) e^{-\beta t}, \quad \beta = \frac{1}{\tau}$$

$$= \frac{1}{2} y_4 \frac{t^2}{\tau t_4} e^{-t/\tau}$$

Step 3: Satisfy the initial Conditions

$$y(0) = y_0$$

$$y = y_h + y_p$$

$$y_{p_1} + y_{p_2} + y_{p_3}$$

$$y(t) = Ce^{-t/\tau} + y_1 \frac{t-\tau}{t_1} + y_2 + y_3 \frac{1}{1-\tau/\alpha} e^{-t/\alpha} + \frac{1}{2} y_4 \frac{t^2}{\tau t_4} e^{-t/\tau}$$

$$y(0) = C - y_1 \frac{\tau}{t_1} + y_2 + y_3 \frac{1}{1-\tau/\alpha}$$

$$C = y_0 + y_1 \frac{\tau}{t_1} - y_2 - y_3 \frac{1}{1-\tau/\alpha}$$

b) Write the dimensions (units) of all variables in terms of dimensions of y and T

$$\tau \frac{dy}{dt} + y = y_1 \frac{t}{t_1} + y_2 + y_3 e^{-t/\alpha} + y_4 \frac{t}{t_4} e^{-\beta t}$$

$$[t][y/t] + [y] = [y][t/t] + [y] + [y][.] + [y][t/t][.]$$

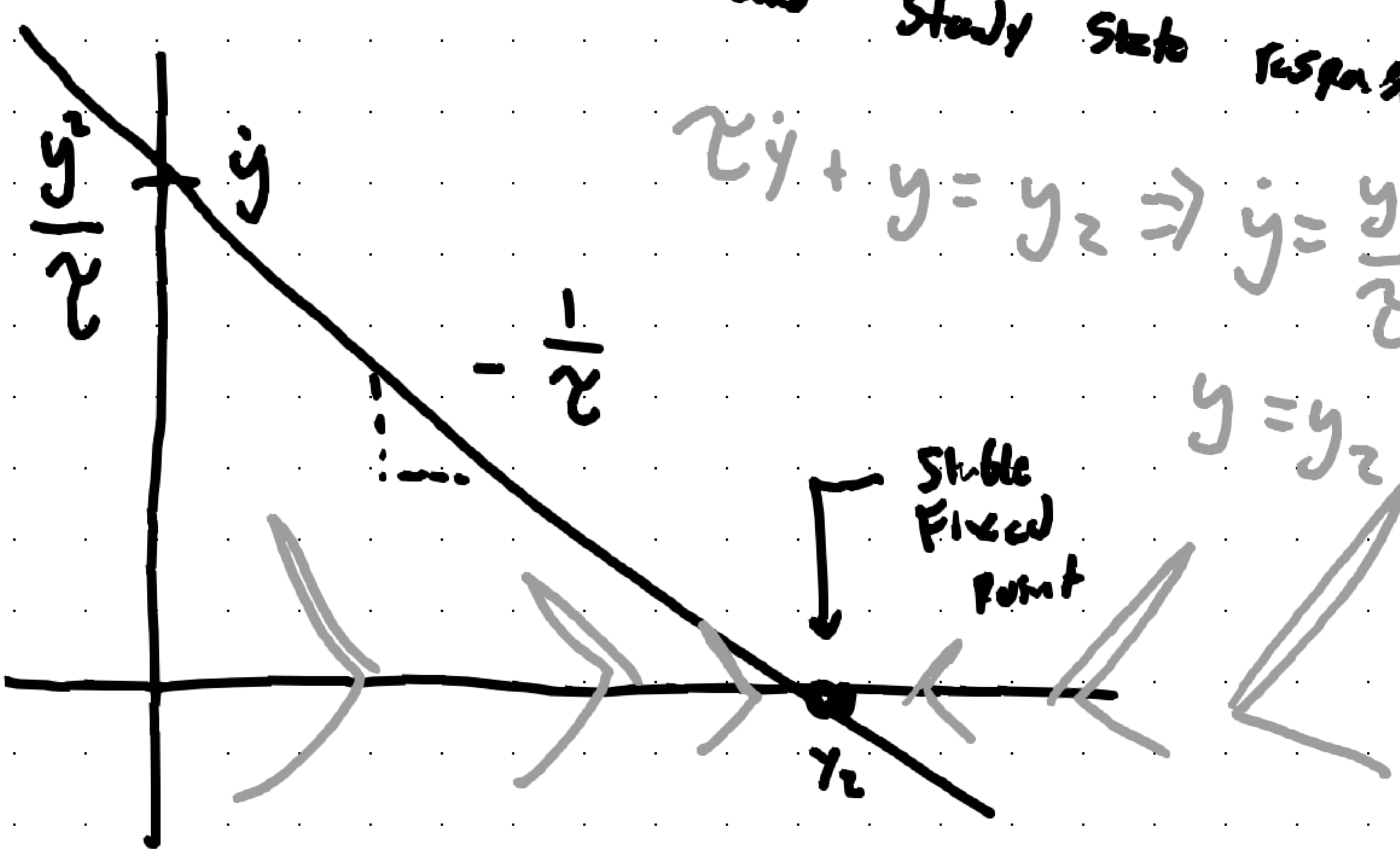
Exponentials have no units!

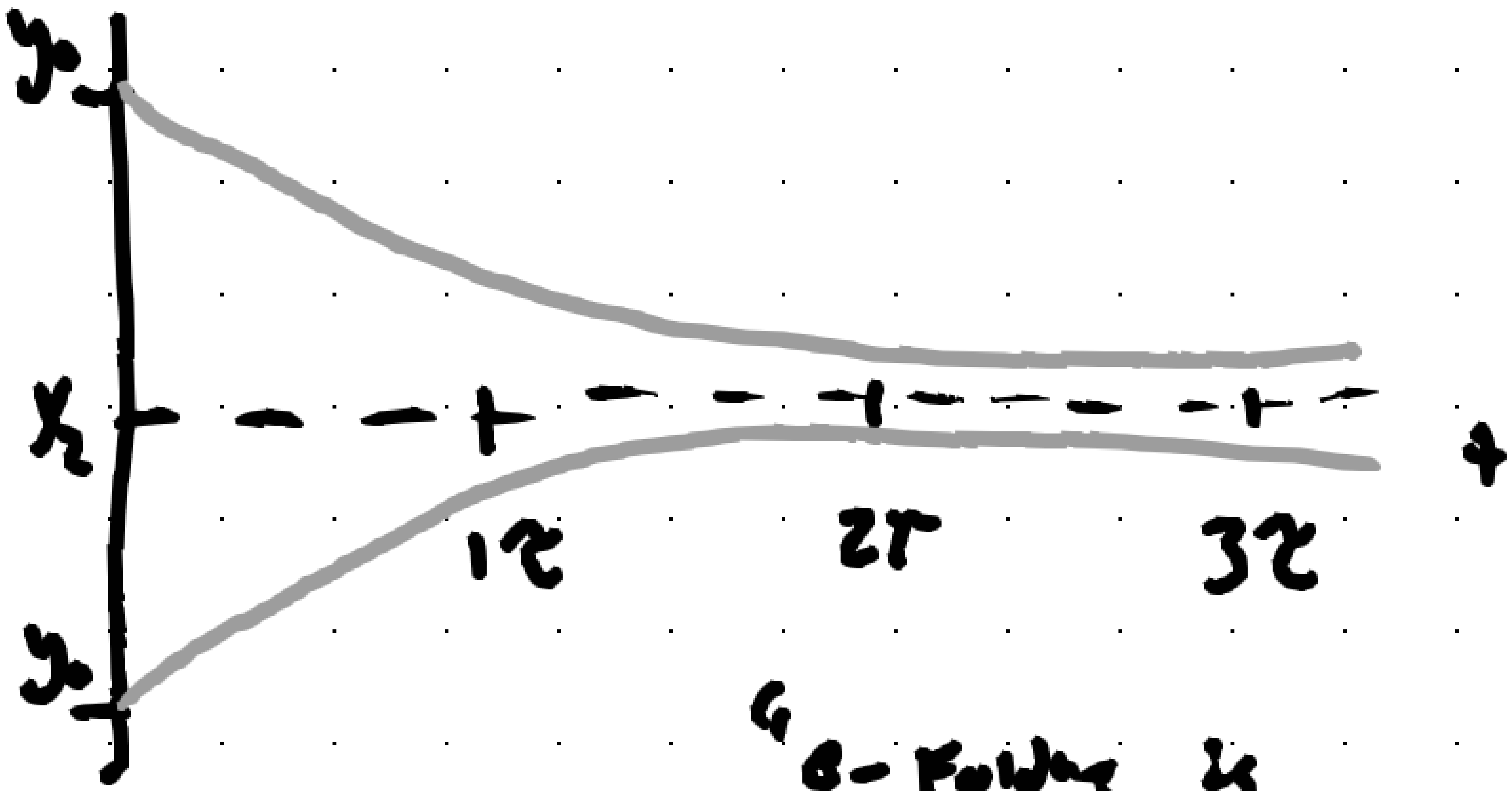
c)

Let $y = y_3 = y_4 = 0$. Sketch the state space and $y(t)$ for $y_2 > 0$, $y_0 > y_2$ and $y_0 < y_2$. Label the transient and steady state responses.

$$\tau \dot{y} + y = y_2 \Rightarrow \dot{y} = \frac{y_2}{\tau} - \frac{y}{\tau}$$

$$y = y_2$$





e-Folding is occuring here!

$$y(t) = \underbrace{(y_0 - y_2)e^{-t/\tau}}_{\text{Transient!}} + \underbrace{y_2}_{\text{Steady State.}}$$

(will be here, but will eventually be gone.)

d) Sketch $y(t)$ for $\tau = t_4 = 1$, $y_0 = 1$ $y_1 = y_2 = y_3 = 0$, $y_4 = 2$,

$$\dot{y} + y = 2t e^{-t}$$

$$y(t) = \underbrace{e^{-t}}_{\text{From } y_h} + \underbrace{t^2 e^{-t}}_{\text{From } y_p} = (1+t^2)e^{-t}$$

(Step One) $(t \ll 1)$ (Small Time)

$e^{☺} \approx (1 + ☺)$

Taylor Series

$$y(t) \approx (1+t^2)(1-t)$$

$$\approx 1 - t + t^2 + t^3$$

$$\approx 1 - t$$

(ii) $(t \gg 1)$ (Big Time!!)

$$y(t) = t^2 e^{-t}$$

(At Big time, t^2 is so much bigger than 1, so we don't care)

(iii) $y(t^*)=0$

$$0=(1+(t^*)^2)e^{-t^*}$$

$$0=1+(t^*)^2$$

$\rightarrow$ No Real Roots (No zero crossings)

(iv) Set Derivative $=0$ to see if it ever flips

Karsten Continues....