Laplace Transformations

$$\mathcal{L}\{f(t)\} = F(S) = \int_0^{\infty} e^{-st} f(t)\,dt$$

$$\boxed{S = \sigma + J\omega}$$

Some Rules:

Linearity

1) $\mathcal{L}\{a\,f(t) + b\,g(t)\} = a\,F(s) + b\,G(s)$

2) $\mathcal{L}\{f'(t)\} = SF(S) - f(0)$

3) $\mathcal{L}\{f''(t)\} = S^2F(S) - Sf(0) - f'(0)$

4) Laplace Transform Table.

Like how $\frac{d}{dt}\sin = \cos$, Laplace has smiller rules!

Examples:

$$y'' + y = 0, \quad y(0) = 1 \quad y'(0) = 0$$

Find Laplace Transform

$$\mathcal{L}(\ddot{y}) + \mathcal{L}(y) = 0$$

$$s^2 Y(s) - s\,\overset{1}{\cancel{Y(0)}} - \overset{0}{\cancel{\dot{y}(0)}} + Y(s) = 0$$

$$Y(s)(s^2 + 1) - s = 0$$

$$\boxed{Y(s) = \frac{s}{s^2+1}}$$

Now, Let's take the inverse Laplace to get back to t!

From Laplace Table:

$$\frac{s}{s^2+B} = \cos bt$$

$$\boxed{y(t) = \cos t}$$

Inverse Laplace

- Informal Laplace (Using the laplace table)

$$f(s) = \frac{3}{s^2+9}, \quad \mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right]$$

* From Table

$$\sinh(at) = \frac{a}{s^2+a^2}$$

$$= \sin 3t$$

- Formal Method (Using partial Fractions)

Example:

* This is trickier! How to decompose?

$$F(s) = \frac{s+3}{(s+1)(s+2)}$$

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

1

For each distinct linear factor (S+a for example) we will assign:

$$\frac{A}{s+a}$$

2

For each repeated linear factor ($(s+a)^n$ for example) we will assign:

$$\frac{A_1}{s+a} + \frac{A_2}{(s+a)^2} + \dots + \frac{A_n}{(s+a)^n}$$

③

For each irreducible factor (such as $s^2 + bs + c$), assign

$$\frac{Bs + C}{s^2 + bs + c}$$

⟶ Multiply through common (initial) denominator & find A, B, C by equating coefficients.

Back to that example from earlier...

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

multiply out

$$s+3 = A(s+2) + B(s+1)$$

$$s+3 = As + 2A + Bs + B$$

s terms $\quad As + Bs = s \rightarrow A + B = 1$

Const $\quad 2A + 1 = 3 \rightarrow A = 2$

So $B = -1$

$$F(s) = \frac{2}{(s+1)} - \frac{1}{(s+2)}$$

From Table

$$F(t) = 2e^{-t} - e^{-2t}$$

Example:

$$F(s) = \frac{s+2}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$$

Multiply out

$$s+2 = A(s+1) + B$$

$$s+2 = As + A + B$$

s terms

$$As = s \qquad A = 1$$

Constants

$$As + B = 2 \qquad B = 2 - s$$

Example 2

$$F(s) = \frac{2s+5}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

→ multiply out

$$2s+5 = A(s^2+4) + (Bs+C)(s+1)$$

$$2s+5 = As^2 + 4A + Bs^2 + Bs + Cs + C$$

s^2 terms:

$$0 = A + D$$

s terms

$$2 = B + C$$

const

$$5 = 4A + C$$

$A = \frac{3}{5}$

$B = -\frac{3}{5}$

$C = \frac{13}{5}$

Cover-Up Method

$$\frac{3}{(s+2)(s-3)} = \frac{A}{(s+2)} + \frac{B}{(s-3)}$$

$$3 = A(s-3) + B(s+2)$$

To solve for each constant, you're going to want to set s equal to something that will cancel out one of the two.

Say solving for A, you'd set $s=-2$, making $B=0$. You can then solve for A.

Solving for B, you'd set $s=3$, making $A=0$. You can now solve for B.

First Shift Theorem

(or exponential Shift Theorem)

$$\mathcal{L}\{f(t)\} = F(s)$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

For example, in the case of $\mathcal{L}(e^{2t}\cos 3t)$

$$\mathcal{L}\{e^{2t}\cos 3t\} = \frac{s-2}{(s-2)^2+9}$$

s

s

This is the Shift! We can call $(s-2)$, s!

Another example:

$$\mathcal{L}\{\sin 5t) = \frac{5}{s^2+25}$$

$$\mathcal{L}\{e^{-4t}\sin 5t) = \frac{5}{(s+4)^2+25}$$

e shifts the s!

Completing the Square

$as^2 + bs + c$

$c = \left(\frac{b}{2}\right)^2$

$$F(s) = \frac{1}{(s^2 + 2s + 2)} = \frac{1}{\underbrace{(s^2 + 2s + 1)}_{(s+1)^2} + 1}$$

$$F(s) = \frac{1}{(s+1)^2 + 1} = e^{-t} \sin t$$

If we had say $\frac{\tau K_0}{(s+1)^2 + 1}$, factor τK_0 out!

Example:

$$F(s) \quad \frac{s}{s^2+2s+5} = \frac{s}{(s^2 + 2s + 1) + 5}$$

$$\left(\tfrac{1}{2}b\right)^2 \quad 1^2$$

$$\Rightarrow F(s) = \frac{s}{(s+1)^2 + 5}$$

Can't do first shift here because of numerator!
Here's what we'll do here though...

$$F(s) = \frac{(s+1)-1}{(s+1)^2 + (\sqrt{5})^2} = \frac{s+1}{(s+1)^2+(\sqrt{5})^2} - \frac{1}{(s+1)^2+(\sqrt{5})^2}$$

We want b^2 for the form

$$e^{-t}\cos(\sqrt{5}) \quad - \frac{\frac{\sqrt{5}}{\sqrt{5}}}{(s+1)^2+(\sqrt{5})^2}$$

← Want to have B!

$$- \frac{1}{\sqrt{5}} \frac{\sqrt{5}}{(s+1)^2+(\sqrt{5})^2}$$

$$- \frac{1}{\sqrt{5}} e^{-t} \sin(\sqrt{5})$$