June 10th, Vector Calculus, Summer

Green's Theorem:

$$\oint_C \vec{f} \cdot T \, ds = \text{Circulation}$$

$\vec{f} = 2D$ $T = $ Unit Tangent vector

- Circulation is a fancy term for work along a closed curve.

- Green's Theorem can also be written as

$$\iint_R \underbrace{(\nabla \times f)}_{\text{Curl of } f} \cdot \hat{K} \, dA$$

Micro Circulation

Region enclosed by C

Recall, we can also write these in different forms...

- $\vec{r}(t)$ = parametric description of C in 2D
- $\vec{F} = (P, Q)$

Green's Theorem has many re-writings!

- $$\oint_C f \cdot \hat{T}\, ds = \oint_c \vec{f} \cdot \frac{dr}{dt}\, dt$$

also

$$\oint_C f \cdot dr \quad \text{and} \quad \oint P\,dx + Q\,dy$$

$$\iint_R (\nabla \times f) \cdot \hat{k}\, dA = \iint_R \left(\frac{df}{dx} - \frac{df}{dy}\right) dA$$

Types of Green's Theorems

Circulation

$$\oint_C \vec{f} \cdot \hat{T}\, ds = \iint_R (\nabla \times \vec{f}) \cdot \hat{k}\, dA$$

Flux

$$\oint_C \underbrace{\vec{f} \cdot \hat{N}}\, ds = \iint_R (\nabla \cdot \vec{f})\, dA$$

Flux out of the closed curve C

Note: These two forms are equivalent!

Why do we care about Green's Theorem?

- Elegance, and Cleanlyness
- Helps give intuitive feel for curl and divergence.
- Mainly, it gives us more options to solve for something like work, circulation, or flux.

More Tools!

Ex: Find $\oint \vec{v} \cdot \hat{T}\, ds$

$v = (x-y, x)$

C = unit circle

Before we do anything, we should check if this field is conservative! Curl must be zero.

$$\nabla \times \vec{v} = \left(\frac{dv_y}{dx} \cdot \frac{\partial v_x}{dy} \right) \hat{K} = (1-(-1))\hat{K}$$

$= 2K$

not conservative

use form $\oint_C \vec{v} \cdot \hat{T}\, ds$

$= \oint \vec{v} \cdot \frac{dr}{dt}\, dt$

Need to define $\vec{r}$ of unit circle!

$0 \le t \le 2\pi$

$\vec{r}(t) = \cos t, \sin t$ → circle with radius of one.

$\frac{dr}{dt} = (-\sin t, \cos t)$

Now, we can plug in

$$\oint_C \overset{v(t)}{(x-y, x)} \cdot \overset{\frac{dr}{dt}}{(-\sin t, \cos t)}\, dt$$

Josh finishes this integral in his notes

OR use green's theorem

The curl we already found!

$$\oint v \cdot T\, ds = \iint_R \underbrace{(\nabla \times \vec{v})}_{2\hat{k}} \cdot k\, dA$$

$$= \iint_R 2\, dA$$

$$= 2 \underbrace{\iint_R dA}$$

This is the area of R

$= \pi R^2$

$= \pi (1)$

$$= 2(\pi)$$

$$= 2\pi$$

Josh has another example
here in his notes

Ex $\vec{v} = (y^2, x^2)$

Josh Does ANOTHER example

EX:

Find the flux out of C

$\oint \vec{F} \cdot N \, ds$

flux form of green's Theorem.

Check for incompressibility!

Path independence!

- $ds = \left\| \frac{dr}{dt} \right\| dt$

speed = distance time

Surfaces and fields

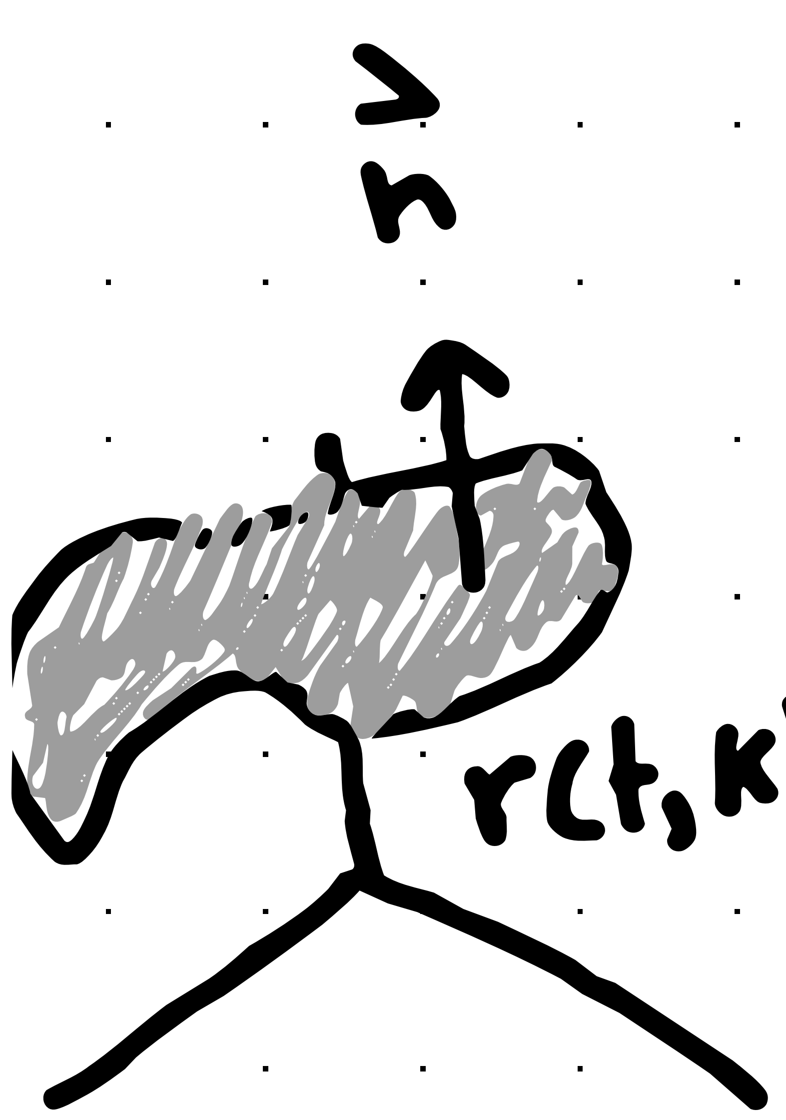
- Our Curves in 3D are Known as Surfaces!

Surfaces are often defined as

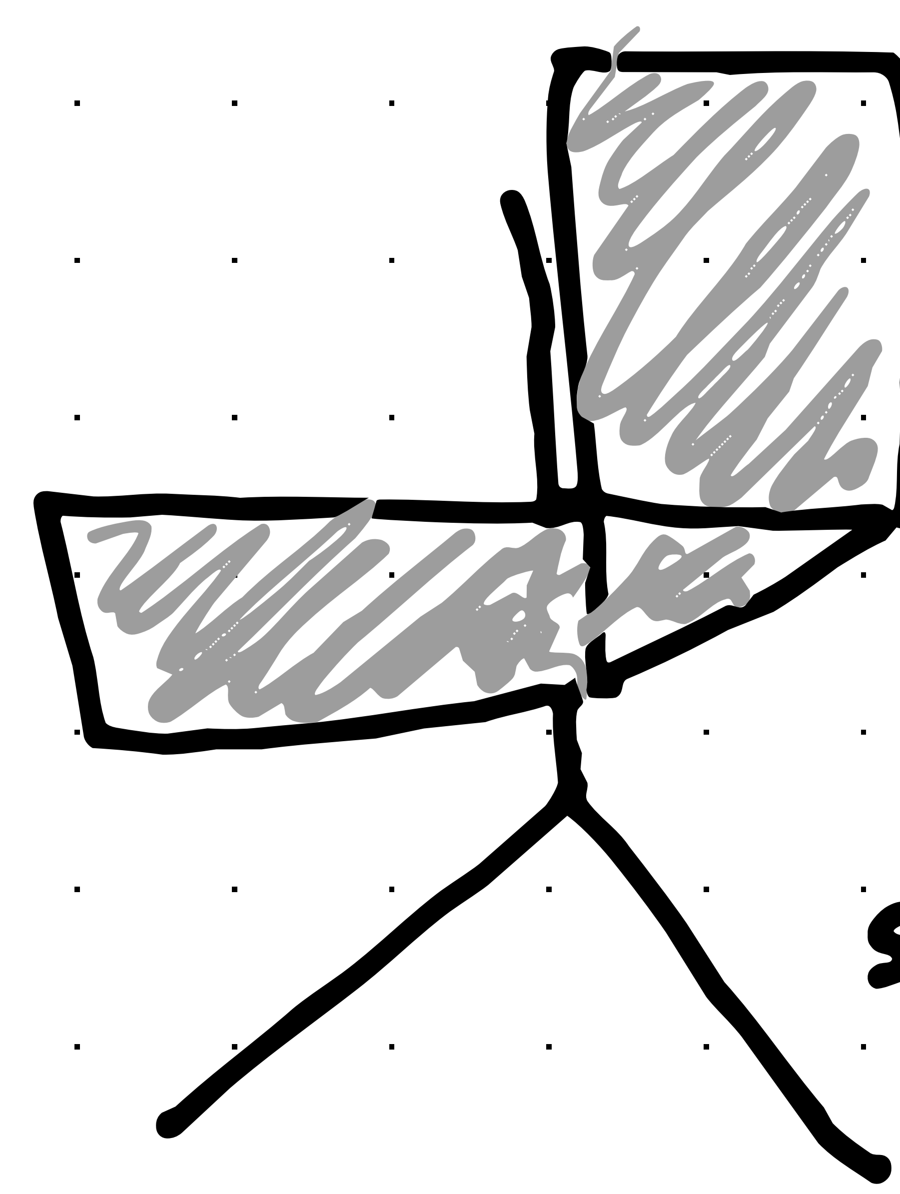
$$z = f(x, y) \quad \text{or}$$

$$F(x, y, z) = 0 \quad \text{or}$$

$$r(t, K) = (x(t, K), y(t, K), z(t, K))$$

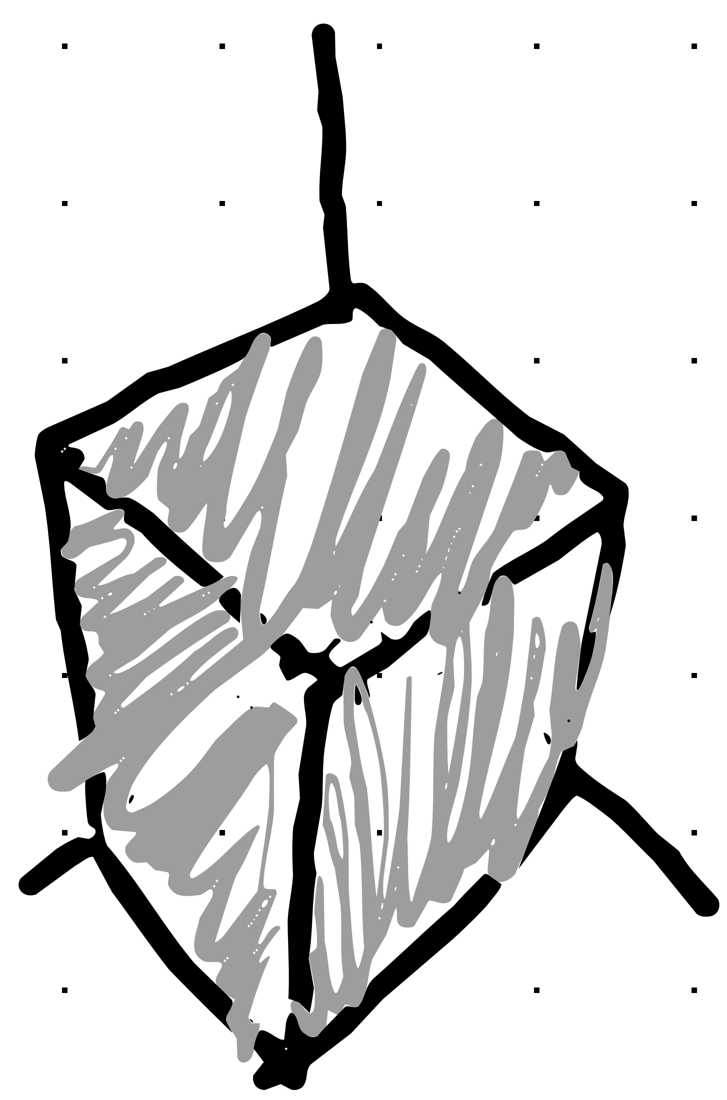


- Smooth curve
- ∇F is normal to the surface



Piecewise smooth curve

Closed Smooth Surface (Spherical Shell)



Piecewise Smooth Closed Surface (Cube)

$\hat{n}$ = normal to surface

Bounded Surfaces

$\hat{n}$ = positive unit upwards (k component is +)

Closed Surfaces

$\hat{n}$ = positive unit pointing out

Recall $F(x, y, z) = 0$ for a surface

$\Rightarrow \nabla f$ is orthogonal to surface $= \hat{n}$

$r(t, k) \Rightarrow \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dk} = \hat{n}$

surface

$$\boxed{\text{Josh Fishman notes on Surfaces and \underline{Surface Integrals}}}$$

PLEASE READ!