Surface Integrals

Josh Begins his day by discussing Finding ds for surface integrals!

PLEASE READ

There is a proof to say...

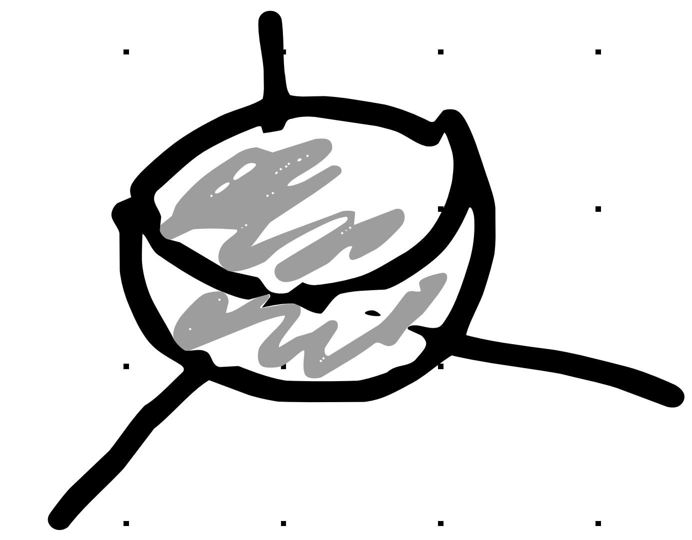
In terms of t and k

$$\boxed{ds = \|\vec{n}\|\, dt\, dk}$$

$$ds = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\; dx\, dy$$

In x and y

Ex find $\iint_S f(x,y,z)\,dS$



$f(x,y,z) = 8 - z$ ← field

Surface → $S \Rightarrow z = (x^2+y^2)^{1/2}, \quad 0 \le z \le 4$

1. Describe S parametrically

$z = (x^2+y^2)^{1/2}$

$r(x,y) = (x, y, (x^2+y^2)^{1/2})$

2. Find F on S

$f(x,y,z) = 8 - z$

$8 - (x^2+y^2)^{1/2}$ ← we'll call this $g(x)$

3. Find ∂S in terms of ∂A

$$\partial S = \|\vec{n}\| \, \partial A$$

$$\vec{n} = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right)$$

* look at original equation for F, and bring the z over....

$$z = (x^2+y^2)^{1/2}$$

$$\frac{\partial z}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}(2x) = \frac{x}{(x^2+y^2)^{1/2}}$$

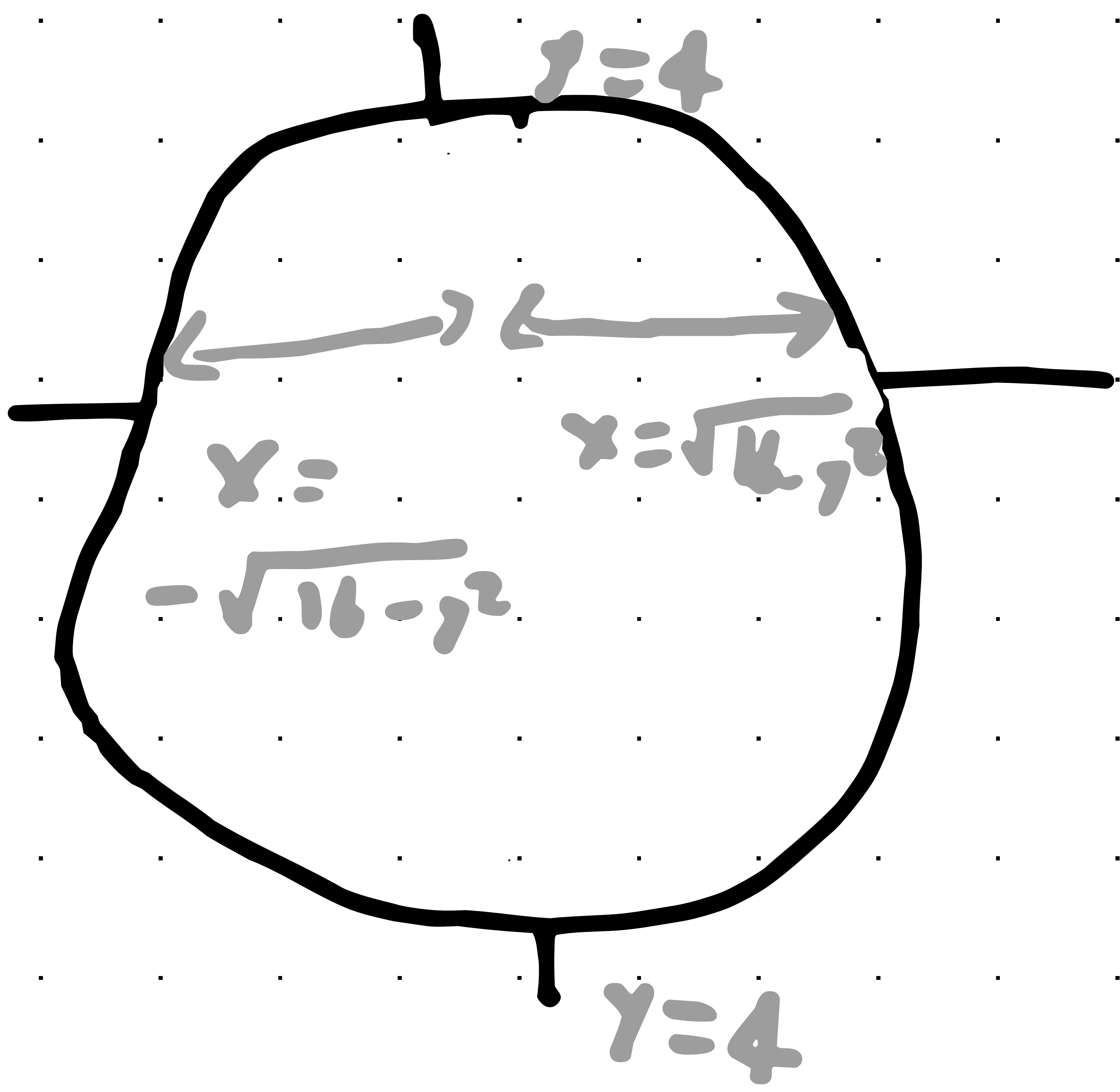
$$\frac{\partial z}{\partial y} = \frac{y}{(x^2+y^2)^{1/2}}$$

$$\|\vec{n}\| = \sqrt{\left(\frac{-x}{(x^2+y^2)^{1/2}}\right)^2 + \left(\frac{-y}{(x^2+y^2)^{1/2}}\right)^2 + 1^2}$$

$$\|\vec{n}\| = \sqrt{z}$$

4. Find limits $\Rightarrow$ range of x & y for R

R $\Rightarrow$ $x^2 + y^2 \le 16$



$-\sqrt{16-y^2} \le x < \sqrt{16-y^2}$

$-4 \le y \le 4$

5. Put it all together

$$\iint_S f\,dS = \iint_R \overbrace{8-(x^2+y^2)^{1/2}}^{g(x,y)}\,\overbrace{\sqrt{2}}^{\|\vec{n}\|}\,\underbrace{dx\,dy}_{dA}$$

Let's convert this to polar, because I don't want to deal with this

Polar => $\quad x = r\cos\theta \quad y = r\sin\theta$

$dA = r\,dr\,d\theta$

Surface Integrals for Flux

$$\iint_S f\,dS$$

Special case where f (thing in the integral) will be $f = \vec{F}\cdot\hat{N}$

Some vector field

Unit normal to surface.

Component F that is orthogonal

$$\text{Flux} = \iint_S \vec{F}\cdot\hat{N}\,dS$$

$$\hookrightarrow \iint_S F\cdot\frac{\vec{n}}{\cancel{\|n\|}}\,\cancel{\|n\|}\,dA$$

$$= \iint_R \vec{F}\cdot\vec{n}\,dA$$

If the field is given as

(P, Q, R)

and we have the normal

$\vec{n} = (n_x, n_y, n_z)$

$$\iint_R (P n_x + Q n_x + R n_z)\, dA$$

Example

Find the upwards Flux of field

$F = (x^2+1, \ yx, \ e^x + zy)$

across the surface $z = \sqrt{9-x^2-y^2}, \ z \geq 0$

$$\text{Flux} = \iint_S F \cdot \hat{N} \, ds = \iint_R \vec{F} \cdot \vec{n} \, dA$$

$$\vec{n} = \left(-\frac{dz}{dx}, -\frac{dz}{dy}, 1\right)$$

$$\frac{dz}{dx} = \frac{-x}{(9-x^2-y^2)^{1/2}} = \frac{-x}{z}$$

$$\frac{dy}{dz} = \frac{-y}{(9-x^2-y^2)^{1/2}} = \frac{-y}{z}$$

$$\hat{n} = \left(-\frac{dz}{dx}, -\frac{dz}{dy}, 1\right)$$

$$\hat{n} = \left(\frac{x}{z}, \frac{y}{z}, 1\right)$$

$$F \cdot \hat{n} =$$

$$(x^2+1, xy, e^x+zy) \cdot \left(\frac{x}{z}, \frac{y}{z}, 1\right)$$

$$= \frac{(x^2+1)y}{z} + \frac{xy^2}{z} + e^x + zy$$

Now, we can plug back in for z

$$\frac{(x^2+1)y}{\sqrt{9-x^2-y^2}} + \frac{xy^2}{\sqrt{9-x^2-y^2}} + e^x + zy$$

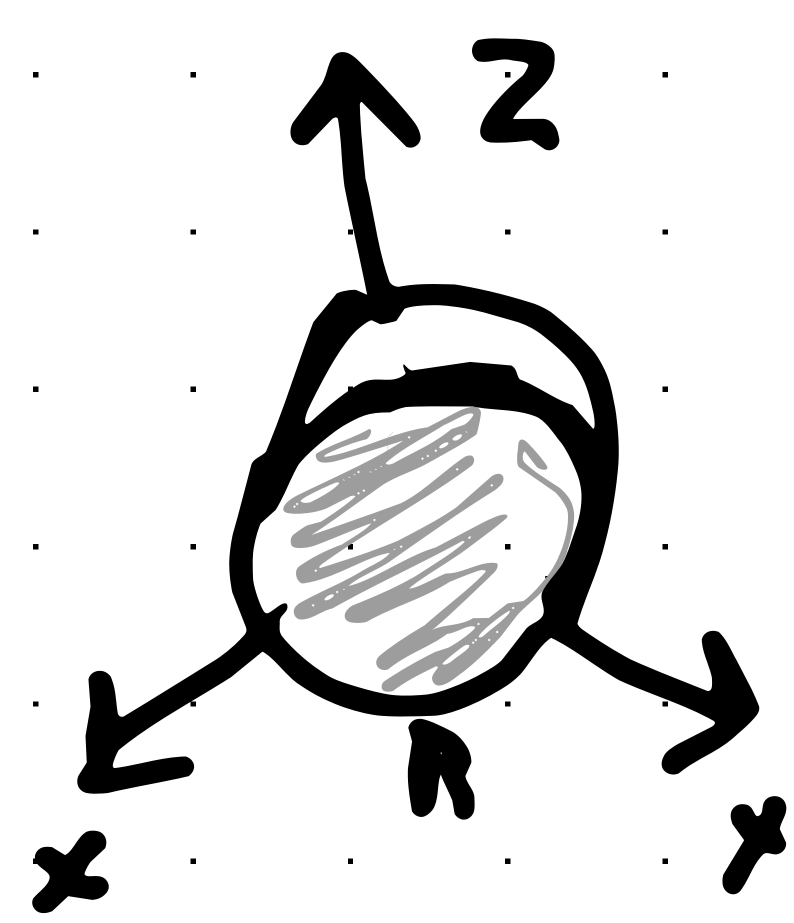
Now, we can put it all together

$$\iint_R \left(\left(\frac{(x^2+1)x + xy^2}{\sqrt{9-x^2-y^2}} \right) + e^x + 2y \right) dx\,dy$$

Now, what is R?

$$z = 9 - x^2 - y^2$$

$$x = \sqrt{9-y^2}$$



when $z=0$

$$0 = \sqrt{9-x^2-y^2}$$

$$x^2+y^2 = 9$$

* circle of radius 3

Terrible drawing, but it is a semi-circle.

$$\int_{y=-3}^{y=3} \int_{x=-\sqrt{9-y^2}}^{x=\sqrt{9-y^2}} \left(\left(\frac{(x^2+1)x + xy^2}{\sqrt{9-x^2-y^2}} \right) + e^x + 2y \right) dx\,dy$$

This is a horrid integral for solving for flux, but is correct. Josh would never ask this us on a test.

Stoke's Theorem

Green's but in 3D!

$$\text{Flux} = \iint_S \vec{F} \cdot \hat{N} \, dS$$

Special case of the surface integral.

+

Special case of $F =$

$\vec{F} = \nabla \times G$

$\nabla \cdot \vec{F} = 0$

F is incompressible

$$\underbrace{\iint_S (\nabla \times G) \cdot \hat{N}\, ds}_{\text{Stoke's Theorem!}} = \overbrace{\oint_C \vec{G} \cdot \hat{T}\, ds}^{\text{Circulation}}$$

C: Boundary of S

Note: If S = flat region in xy plane

$ds = dA \qquad \hat{N} = \hat{k}$

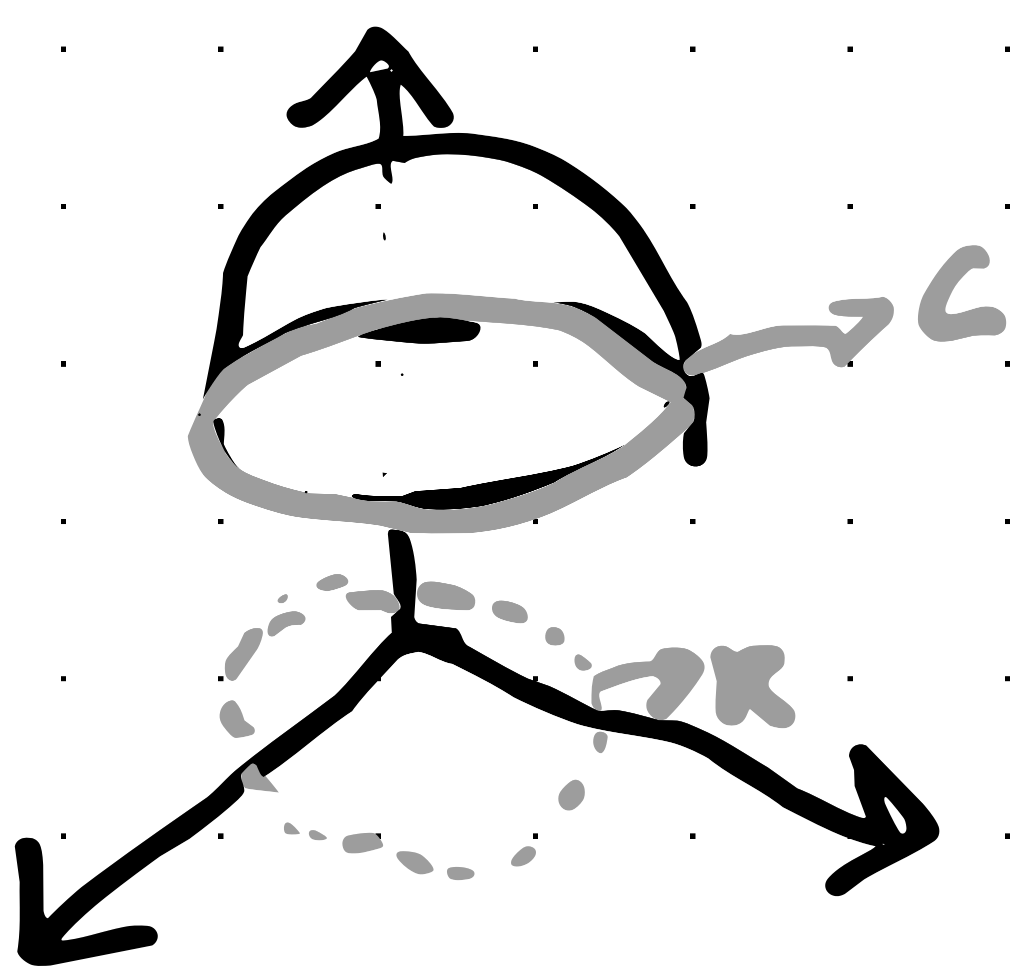
$$\Rightarrow \iint_R (\nabla \times G) \cdot \hat{k}\, dA = \oint_C \vec{G} \cdot \hat{T}\, ds$$

This is just Green's Theorem.

Ex. Find $\iint_S (\nabla \times F) \cdot \hat{N} \, ds$

$F = (-xz, yz, xye^z)$

$S \Rightarrow z = 5 - x^2 - y^2 \qquad z \geq 3$



Floating upside down bowl

Method 1: Evaluating the Surface Integral

$\longrightarrow$ Calculating $\nabla \times F$

$\longrightarrow$ Calculating $\hat{n}$

$\longrightarrow$ $(\nabla \times F) \cdot \hat{n}$

$\longrightarrow$ do the double integral over R

* gross and hard

OR Method 2: Stokes Theorem

$$= \oint_C F \cdot \hat{T} \, ds$$

$C = r(t) =$ boundary of surface

$C = \; 3 = 5 - x^2 - y^2$

$2 = x^2 + y^2$ ← circle of radius $\sqrt{2}$

$\vec{r}(t) = (\sqrt{2}\cos t, \sqrt{2}\sin t, 3)$

$\frac{d\vec{r}}{dt} = (-\sqrt{2}\sin t, \sqrt{2}\cos t, 0)$

So now, we just treat it like a regular line integral....

$$\oint \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt$$

$$F \cdot \frac{d\vec{r}}{dt} =$$

$$(-xz,\ yz,\ xye^{z}) \cdot (-\sqrt{2}\sin t,\ \sqrt{2}\cos t,\ 0)$$

$$= \sqrt{2}\, xz \sin t + \sqrt{2}\, yz \cos t$$

Plug in for x, y, z now

$$\sqrt{2}(\sqrt{2}\cos t)(3)\sin t + \sqrt{2}(\sqrt{2}\sin t)(3)\cos t$$

from r(t), we know that t maxes out at 2π. (cos and sin max at once)

$$= 12\cos t \sin t = \underline{6\sin 2t}$$

$$\int_{t=0}^{2\pi} 6\sin 2t\, dt = -\frac{6}{2}\cos 2t \Big|_{0}^{2\pi} = 0$$